ABSTRACT

MODIFIED COMMUTATION RELATIONSHIPS FROM THE BERRY-KEATING PROGRAM

This work is a study of the introduction of gravity to quantum theory. The presence of the Riemann zeta function in string theoretic arguments suggests a connection between the zeta function and quantum theory. Current approaches to quantum gravity suggest there should be a modification of the standard quantum mechanical commutator, $[x, p] = i\hbar$. Typical modifications are phenomenological and designed to result in a minimal length scale. As a motivating principle for the modification of the position and momentum commutator, we assume the validity of a version of the Bender-Brody-Müller variant of the Berry-Keating approach to the Riemann hypothesis. Through our research we arrive at a family of modified position and momentum operators, and their associated modified commutator, which leads to a minimal length scale. Additionally, this larger family generalizes the Bender-Brody-Müller approach to the Riemann hypothesis.

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MODIFIED COMMUTATION RELATIONSHIPS FROM THE BERRY-KEATING PROGRAM

by

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To my family, friends, and mentors, for always reminding me how I am make a positive contribution to their lives.

INTRODUCTION

In this work, the validity of the Riemann Hypothesis is assumed and applied to a mathematical framework which suggests quantum gravity should introduce some non-zero minimal distance scale Δx_0. Although there is no agreed upon approach to quantizing gravity, string theory based arguments lead to such a minimum absolute length [1]. This result could potentially solve renormalizability, one of the major problems in perturbative quantum mechanics. It has been suggested in many works [1, 2, 3, 4, 5, 6] how a modification of the standard quantum commutator, $[\hat{x}, \hat{p}] = i\hbar$, could induce a lower bound to the possible resolution of distances.

Motivated by the Bender-Brody-Müller approach to the Riemann Hypothesis [8], we propose a modification to the standard quantum mechanical operators $\hat{x}$ and $\hat{p}$ so that they follow a similar mathematical foundation. Bender, Brody and Müller proposed the following operator which satisfies the Riemann hypothesis condition:

$$\hat{H} = \hat{\Delta}_{BBM}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}_{BBM} \tag{1}$$

where $\hat{\Delta}_{BBM} = 1 - e^{-i\hat{p}}$. Motivated by this "Hamiltonian", we develop a group of similar "Hamiltonians", which lead to a family of modified position and momentum operators and their modified commutators is investigated. The modified operators are symmetric and to lead to a minimum length scale similar to those found in Kempf, Mangano, and Mann [7]. The major difference in our approach is the modification of both position and momentum operators.

RIEMANN HYPOTHESIS

Bernhard Riemann, a German mathematician, published a short 8-page paper in 1859 in which he estimated the number of primes less than a given magnitude using a certain *meromorphic function* on $\mathbb{C}$ [9]. A meromorphic function is a function that is defined on an open subset D of the complex plane; and except for a set of isolated points, referred to as poles, is complex differentiable around every point within its domain [10]. This function was an elaboration of an idea first proposed by Carl Friedrich Gauss [10], who conjectured in 1792 that the number of primes less than or equal to an integer "n" asymptotically approaches $\frac{1}{\log(N)}$ [10]. The aforementioned meromorphic function is known today as the Euler-Riemann zeta function [9].

Any whole number can be factored to its prime constituents. This is why prime numbers are the fundamental building blocks of numbers and have unique properties [11]. For instance, an arbitrarily large number can be produced by multiplying any number of primes together. But if one wishes to factor an arbitrarily large number to its prime constituents, the process is much more complicated. Prime factorization is the basis of modern computer cryptography and one of the many reasons why prime numbers remain of great interest [12]. Finding an accurate model for the distribution of prime numbers is one of the greatest unsolved problems in modern mathematics. However, thanks to Riemann's paper, progressive steps were taken in providing such a model since the zeta function can be presented as a product of prime numbers. The prime number theorem describes the distribution of prime numbers [13]. Prime number theory is based on Gauss' idea that prime numbers should become less common as they become larger. The theorem was proved in 1896 by Jacques Hadamard and Charles Jean de la Vallee Poussin with help from the ideas of Riemann, namely the zeta function [13]. Their proposed prime counting function is commonly referred to as the pi-function and has the form:

$$\pi(N) \sim \frac{N}{\log(N)} \tag{2}$$

where N is some real positive number and $\pi(N)$ is the number of primes less than or equal to N. Equivalently, we can write:

$$\pi(N) = \sum_{p \leq N} 1, \tag{3}$$

that is, the number of primes less than or equal to N. If we apply a weight of $\log(N)$ to every prime we obtain the estimate:

$$\theta(N) = \sum_{p \leq N} \log(N) \sim N \tag{4}$$

which directly implies that the average gap between consecutive prime numbers less than or equal to N is approximately $\log(N)$.

Since all natural numbers are either prime or composite, it is always possible to factor a composite number into a product of prime numbers (prime factorization). Euclid's Theorem is an assertion that there are infinitely many primes [14]. The proof of Euclid's theorem goes as follows:

"Assume that the set of prime numbers is not infinite. Make a list of all the primes. Next, let P be the product of all the primes in the list (multiply all the

primes in the list). Add 1 to the resulting number, Q = P +1. As with all numbers, this number Q must be either prime or composite:

- *If Q is prime, then a prime that was not originally in the list has been found.*

- *If Q is not prime, it is composite, i.e. made up of prime numbers, one of which, p, would divide Q (since all composite numbers are products of prime numbers). Every prime p that makes up P obviously divides P. If p divides both P and Q, then it would have to also divide the difference between the two, which is 1. No prime number divides 1, and so the number p cannot be on your list, another contradiction that your list contains all prime numbers. [15]"*

Euclid formulated this proof by considering the effect of multiplying together the set of prime numbers . This is interesting as the zeta function can be represented as an infinite product of all the prime numbers. This connection has lead to the idea that any successful prime counting function must also include an infinite sum or series in some way [14].

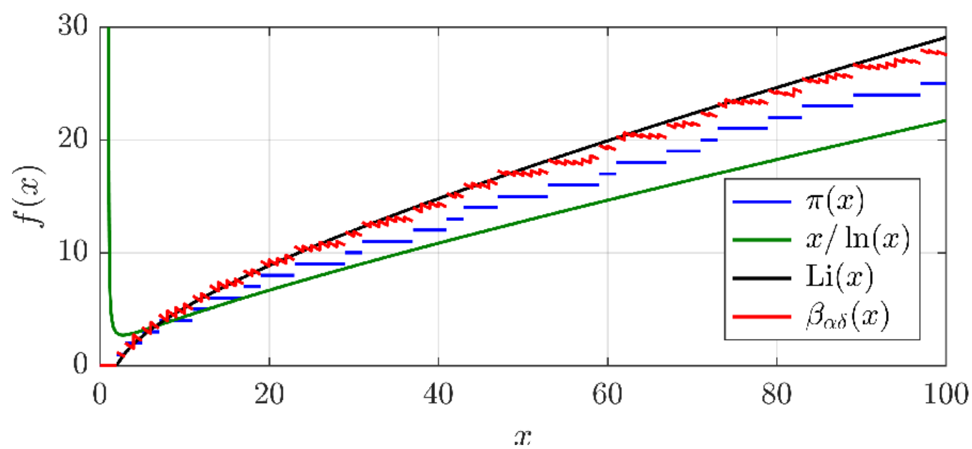

Figure 1. Graphical representation of the Pi function. [16]

The Pi function counts the number of primes up to a given magnitude. As $x \to +\infty$, it appears that $f(x) \to +\infty$ which implies a limitless number of primes. The red approximation is derived from a specific additive function discussed by Vartziotis and Wipper [16].

Dirichlet Series

Peter Gustav Lejeune Dirichlet, a German mathematician, made deep contributions to number theory and Fourier series [10]. Dirichlet developed a series that could be used as a generator for counting weighted sets of objects [10]. The development of the Euler-Riemann zeta function begins by defining this series as the general Dirichlet series, which is an infinite series of the form:

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{5}$$

where a_n, s are complex numbers and λ_n is a non-negative sequence of real numbers that tends to infinity [10]. The next step in developing the Zeta function is to consider a special case of the general Dirichlet series. A simple Dirichlet series is one in which we make the substitution $\lambda_n = \ln(n)$.

$$\sum_{n=1}^{\infty} a_n e^{-\ln(n)s} = \sum_{n=1}^{\infty} a_n e^{-\ln(n^s)} = \sum_{n=1}^{\infty} a_n n^{-s} \tag{6}$$

The Dirichlet series in this form is commonly used in analytic number theory as well as the investigation of the distribution of prime numbers. For more details on how equation (6) relates to the prime number theorem, see reference [10]. If we then let $a_n = \chi(n)$ which is an arithmetic function meaning it is defined for all positive integers n and complex-valued, (known as the Dirichlet character), and we restrict the complex number s to have a real part greater than 1, $Re(s) > 1$, the resulting series is called the Dirichlet L-Series:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{7}$$

The Dirichlet character is a function that is not identically equal to zero, and acts as a structure-preserving map between two algebraic structures [17]. These functions are generalized forms of the Riemann zeta function. To obtain the exact Riemann zeta function we must consider $\chi(n) = 1$. The sum is conventionally represented in integral form as seen below in equation (8). Both representations converge and define an analytic function when $Re(s) > 1$. A change of variable from s to z and $L(s, \chi)$ to $\zeta(z)$ is the final step in constructing the recognizable zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{\Gamma(z)} \int_0^{\infty} dt \frac{t^{z-1}}{e^t - 1}. \tag{8}$$

By expanding the denominator of the integrand in the form of a geometric series, the integral can be rewritten as the series on the left. A specific value of interest would be when $z = 1$, but at this value both representations will diverge. Because $z = 1$ is a zero for $1/\zeta(z)$ it is known as a simple pole of $\zeta(z)$.

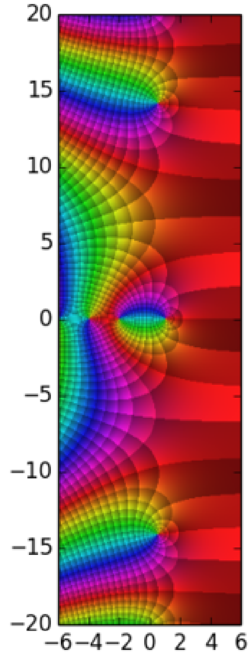

Figure 2. Graphical representation of the Riemann zeta function. [18]

Domain coloring is employed to represent the complex function on two real axes. A color is assigned to each point on the complex plane. Further investigation of the zeta function leads to a direct connection to prime numbers in the following way:

$$\zeta(z) = \prod_{p} \frac{1}{1 - p^{-z}}, \tag{9}$$

where p is every prime number starting at 2. This identity was proven by Leonhard Euler and is known as the Euler product formula for the Riemann zeta function. If we restrict $Re(z) > 1$, the following proof is valid.

Begin by multiplying both sides of the zeta function by its lowest fractional term. Incidentally, that first term also contains the first prime number.

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} + ... \tag{10}$$

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z} + ... \tag{11}$$

By subtracting equation (11) from (10) to remove all elements that have a factor of 2, now the lowest fractional term on the right-hand side contains the second prime number

$$\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + ... \tag{12}$$

Repeating the process for the next term:

$$\frac{1}{3^z}\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{9^z} + \frac{1}{15^z} + \frac{1}{21^z} + ... \tag{13}$$

Subtracting again from equation (12).

$$\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \frac{1}{13^z} + ... \tag{14}$$

Repeating iteratively such that all fractional terms are eliminated from the right-hand side and a product of numbers contain each prime remains on the left-hand side.

$$...\left(1 - \frac{1}{11^z}\right)\left(1 - \frac{1}{7^z}\right)\left(1 - \frac{1}{5^z}\right)\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 \tag{15}$$

Dividing both sides by everything except $\zeta(z)$.

$$\zeta(z) = \frac{1}{...\left(1-\frac{1}{11^z}\right)\left(1-\frac{1}{7^z}\right)\left(1-\frac{1}{5^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{2^z}\right)} \tag{16}$$

Finally, we can write the result more concisely:

$$\zeta(z) = \prod_{p}^{\infty} \frac{1}{1-p^{-z}}. \tag{17}$$

Now we have constructed the zeta function from the general Dirichlet series and shown the direct connection to prime numbers. However, if we are to employ the zeta function in any meaningful way we must address the inherent domain restrictions. It is clear from the series in equation (8) that the zeta function is no longer well behaved when $z \leq 1$. We extend the domain of the function by means of an analytic continuation.

Analytic Continuation

When dealing with infinite sums, it is often the case that only a certain range of values will lead to convergence. The importance of convergence is simply that divergent series cannot be used to make predictions as they do not have specified values. Therefore, if we intend to extract any useful information from an infinite series, it must be convergent or asymptotic. Analytic continuation is a method of expanding the domain of a function or series such that it is well-defined over a larger set of values [13]. The simplest example of analytic continuation begins with the infinite geometric series:

$$S_n = \sum_{n=0}^{\infty} x^n. \tag{18}$$

It is obvious that if the value of x is a positive integer greater than 1, the sum will diverge to infinity as $n \to \infty$. However, if we express the sum in terms of the second and last terms:

$$1 + xS_n = 1 + x + x^2 + x^3 + ... + x^n + x^{n+1} \tag{19}$$

$$1 + xS_n = S_n + x^{n+1} \tag{20}$$

Then solve the above equation for S_n:

$$S_n = \frac{x^{n+1} - 1}{x - 1}. \tag{21}$$

We see that if the magnitude of x is less than 1, the quantity x^{n+1} goes to zero as $n \to \infty$. We are left with an analytic continuation of the geometric series

$$S = \frac{1}{1 - x}. \tag{22}$$

If the value of x in our analytically continued function takes on a value of a positive even integer such as 4, the series does diverge to infinity but its analytic continuation instead assumes a value of $\frac{-1}{3}$. There is still a domain restricted value of $x = 1$.

In the case of the Riemann zeta function, we can begin by viewing the graph of the function on the complex plane as seen below in Figure 3. Note that complex exponents result in a curving on the output space. This can be seen directly through Eulers Formula, the relationship between trigonometric functions and complex exponents [17]:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{23}$$

$$x^{in} = e^{\ln(x^{in})} = e^{in(\ln(x))} = \cos(n * \ln(x)) + i\sin(n * \ln(x)). \tag{24}$$

Because we are considering complex inputs to the zeta function, each input results in a unique curve that ends at a specific value. It will be shown later that negative even integers result in curves that end exactly at the origin. These specific values are known as the trivial zeros of the zeta function.

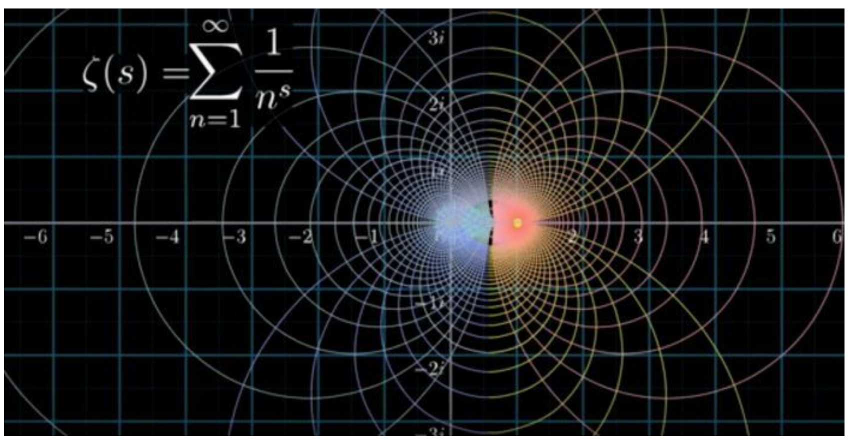

Figure 3. Zeta function in polar coordinates. [19]

The red portion of the graph corresponds to the convergent values of the zeta function where $\mathbb{R}(z) > 1$. The blue portion is a reflection about the line $\mathbb{R}(z) = \frac{1}{2}$. [45]. By employing the symmetry of the zeta function for $\mathbb{R}(z) > 1$, we can reflect the image about the critical line $\mathbb{R} = \frac{1}{2}$. By demanding that the reflected portion have a continuous derivative everywhere, and therefore maintaining the same intersection angles, we end up with only one possible extension of the zeta function [20]. The mathematical model that describes this extension is known as the functional equation:

$$\zeta(z) = 2^z \pi^{z-1} \sin(\frac{\pi z}{2})\Gamma(1-z)\zeta(1-z) \tag{25}$$

So far, it is defined for $\mathbb{R}(z) > 1$ or $\mathbb{R}(z) < 0$, but not in the critical strip where $0 < \mathbb{R}(z) < 1$. A complete derivation of this formula can be found in Arfken and Weber [21].

Non-Trivial Zeroes

Now that we have a mathematical model for the Riemann zeta function, we can begin to consider where the zeros of the function might exist. By making the substitution $z = -2n$ where $n = 1, 2, 3...$ in the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin(\frac{\pi z}{2})\Gamma(1-z)\zeta(1-z), \tag{26}$$

the zeta function vanishes:

$$\zeta(-2n) = 2^{-2n}\pi^{(-2n)-1} \sin\left(\frac{\pi}{2} * (-2n)\right) = \sin(-\pi n) = 0. \tag{27}$$

These values are commonly referred to as the *trivial zeros*. The Riemann hypothesis states that the *non-trivial zeros* will always have a real part equal to one-half, $z = \frac{1}{2} + it$ where $t \in \mathbb{R}$, which has been shown to be true for the first 10^{13} calculated zeros to date [22]. Each zero is known to over 1000 decimal digits each, below are the first five of the calculated non-trivial zeros [18]:

$$\begin{aligned} z_1 &= \frac{1}{2} + i14.13472... \\ z_2 &= \frac{1}{2} + i21.02203... \\ z_3 &= \frac{1}{2} + i25.0108... \\ z_4 &= \frac{1}{2} + i30.42487... \\ z_5 &= \frac{1}{2} + i32.93506... \end{aligned} \tag{28}$$

Despite the large quantity of values that do satisfy the Riemann condition, this assertion has yet to be proven by means of a formal proof.

The importance of Riemann's hypothesis can be explained if we again consider the question: "How many primes are there less than x?". Recalling that a good approximation to this question as stated in equation (2) is about $\frac{x}{log(x)}$, a connection can be drawn from these approximations to the Riemann hypothesis. Firstly, it was Carl Friedrich Gauss who found a function which more accurately predicts the number of primes called the Eulerian Logarithmic Integral [13]:

$$Li(x) = \int_2^x \frac{dt}{\ln(t)}. \tag{29}$$

Secondly, for this to be an accurate approximation over all possible values of x it is important to know the farthest that this $Li(x)$ function ever gets from the actual number of primes less than a given value. The difference between $Li(x)$ and the actual number of primes is often called the error term in Prime Number Theory. It has been found that the error term can at least take on a maximum value of $\sqrt{x} * \ln(x)$ [23]. If the Riemann hypothesis is indeed true, then the error term could be proven to never take on a value greater than $\sqrt{x} * \ln(x)$, which is one reason why proving the Riemann hypothesis would be such a great achievement [23].

MINIMAL LENGTH UNCERTAINTY RELATION

It has been suggested that gravity itself should naturally lead to an effective ultraviolet cutoff and limit to the resolution of space-time [1]. A limit of this sort translates to an absolute minimal length. Present theories explain that high energies used in resolving small distances lead to gravitational effects that disturb the space-time structure. Space-time locality has successfully been probed to the 1 TeV scale though this effect is thought to occur at energies as large as the Planck scale (1.2209×10^{19} GeV). If a lower bound to the possible resolution of distances proves true, then this behavior of gravity could regularize quantum field theories rather than disturbing renormalization. It is an assumption that a minimal length scale should be theoretically described in quantum mechanics as a non-zero minimal uncertainty Δx_0. This assumption is supported by string theoretic arguments that also lead to a minimal length effectively in the form of a minimal position uncertainty [8]. The following arguments are made by considering only the case of minimal uncertainty in position.

Representation on Momentum Space

A modification to the standard quantum mechanical commutator for position and momentum is a method to invoke a minimal uncertainty in position. As suggested by Kempf, Mangano and Mann [1] we start with the following relation:

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \alpha\mathbf{x}^{\mathbf{2}} + \beta\mathbf{p}^{\mathbf{2}}). \tag{30}$$

Where **x** and **p** are modified position and momentum operators. It is a standard result in quantum mechanics that for any pair of observables A, B which are represented as, by the mathematics definition, symmetric operators on a domain of A^2 and B^2, the uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle| \tag{31}$$

holds [24]. By substitution, we find the following:

$$\Delta x \Delta p \geq \frac{1}{2}|\langle (1 + \alpha \mathbf{x^2} + \beta \mathbf{p^2}) \rangle| \tag{32}$$

which leads to

$$\Delta x \Delta p \geq \frac{1}{2}(1 + \alpha \langle \mathbf{x^2} \rangle + \beta \langle \mathbf{p^2} \rangle). \tag{33}$$

Recalling that the uncertainty in a quantum mechanical operator is defined as the square root of the variance:

$$(\Delta \hat{A})^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2. \tag{34}$$

Solving for $\langle \hat{A}^2 \rangle$ and substituting into equation (33) yields the following relation

$$\Delta x \Delta p \geq \frac{1}{2}(1 + \alpha((\Delta x)^2 + \langle x \rangle^2) + \beta((\Delta p)^2 + \langle p \rangle^2)). \tag{35}$$

If we define $\gamma = \alpha \langle x \rangle^2 + \beta \langle p \rangle^2$, we arrive at a general uncertainty relationship

$$\Delta x \Delta p \geq \frac{1}{2}(1 + \alpha(\Delta x)^2 + \beta(\Delta p)^2 + \gamma). \tag{36}$$

By setting α to zero, we can recover from [1] the original argument of only considering the case of a minimal uncertainty in position

$$\Delta x \Delta p \geq \frac{1}{2}(1 + \beta(\Delta p)^2 + \gamma). \tag{37}$$

Where β and γ are positive and independent of Δx and Δp. The new curve in uncertainty takes on the following form in the figure below.

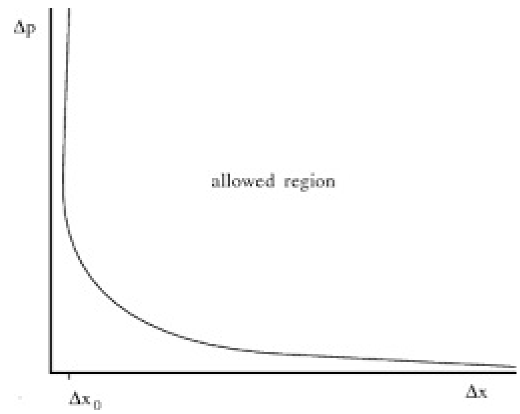

Figure 4. Minimal uncertainty in position [7].

The curve on the boundary of the allowed region is represented by

$$\Delta p = \frac{\Delta x}{\hbar\beta} \pm \sqrt{(\frac{\Delta x}{\hbar\beta})^2 - \frac{1}{\beta} - \langle p \rangle^2}. \tag{38}$$

Traditional uncertainties in position and momentum have the freedom to be as arbitrarily large or small as the other grows correspondingly. In this model, Δx can no longer be made arbitrarily small but instead takes on a specific minimal value which can be solved as follows from

$$\Delta x \Delta p \geq \frac{1}{2}(1 + \beta(\Delta p)^2 + \beta\langle p \rangle^2). \tag{39}$$

Rearranging equation (38) to match the form of a quadratic:

$$\beta(\Delta p)^2 - 2\Delta x(\Delta p) + (1 + \beta\langle p\rangle^2) = 0 \tag{40}$$

Solving the quadratic

$$\Delta p = \frac{\Delta x}{\hbar\beta} \pm \sqrt{(\frac{\Delta x}{\hbar\beta})^2 - \frac{1}{\beta} - \langle p\rangle^2}. \tag{41}$$

It is clear that the minimal value of Δx occurs when the discriminant of the square root is zero.

$$\Delta x_{min}(\langle p\rangle) = \hbar\sqrt{\beta}\sqrt{1 + \beta\langle p\rangle^2} \tag{42}$$

If we consider the absolute minimal value which occurs when $\langle p\rangle = 0$ we have the value

$$\Delta x_0 = \hbar\sqrt{\beta}, \tag{43}$$

which gives us an explicit relationship between the unknown parameter β and the minimal length Δx_0.

Functional Analysis of the Position Operator

Considering the eigenvalue problem for the position operator in momentum space, we have the following differential equation:

$$i\hbar(1 + \beta p^2)\partial_p\psi_\lambda(p) = \lambda\psi_\lambda(p), \tag{44}$$

where λ corresponds to the position eigenvalue. Recognizing the presence of a logarithmic derivative, one can solve the differential equation as follows:

$$\frac{d\psi_\lambda(p)}{\psi_\lambda(p)} = \frac{\lambda dp}{i\hbar(1+\beta p^2)}. \tag{45}$$

Integrating both sides with respect to p and defining the unknown constant as "C" we come to the result:

$$\log\left(\frac{\psi_\lambda}{C}\right) = \int \frac{\lambda dp}{i\hbar(1+\beta p^2)} \tag{46}$$

The integral on the right is

$$\log(\frac{\psi_\lambda}{C}) = \frac{-i\lambda}{\hbar\sqrt{\beta}} \tan^{-1}(\sqrt{\beta}p). \tag{47}$$

The above equation can then be simplified to obtain the formal position eigenvectors.

$$\psi_\lambda(p) = Ce^{\frac{-i\lambda}{\hbar\sqrt{\beta}} \tan^{-1}(\sqrt{\beta}p)} \tag{48}$$

We solve for C such that the eigenstates are normalized

$$1 = CC^* \int_{-\infty}^{+\infty} \frac{1}{1+\beta p^2} = CC^*\pi/\sqrt{\beta}, \tag{49}$$

thus

$$\psi_\lambda(p) = \sqrt{\frac{\sqrt{\beta}}{\pi}} e^{\frac{-i\lambda}{\hbar\sqrt{\beta}} tan^{-1}(\sqrt{\beta}p)}. \tag{50}$$

These eigenvectors cannot be physical states because they do not satisfy the uncertainty relation. From this condition, we can construct the one-parameter family of diagonalizations of x explicitly. For this we calculate the scalar product of the eigenstates for the position operator $|\psi_\lambda\rangle$.

$$\langle\psi_{\lambda'}|\psi_\lambda\rangle = \frac{\sqrt{\beta}}{\pi}\int_{-\infty}^{+\infty}\frac{dp}{1+\beta p^2}e^{-i\frac{(\lambda-\lambda')}{\hbar\sqrt{\beta}}\tan^{-1}(\sqrt{\beta}p)} \tag{51}$$

$$\langle\psi_{\lambda'}|\psi_\lambda\rangle = \frac{2\hbar\sqrt{\beta}}{\pi(\lambda-\lambda')}\sin\left(\frac{\pi(\lambda-\lambda')}{2\hbar\sqrt{\beta}}\right) \tag{52}$$

It is no longer the case that the position eigenstates are orthogonal. However, from equation (53) we can directly read off the family of diagonalizations of x. The sets of eigenvectors parameterized by $\lambda \in [-1,1]$, are comprised of mutually orthogonal eigenvectors

$$\langle\psi_{(2n+\lambda)\hbar\sqrt{\beta}}|\psi_{(2n'+\lambda)\hbar\sqrt{\beta}}\rangle = \delta_{n,n'}. \tag{53}$$

There exist diagonalizations of the position operator, however they are not physical states. They do not exist in the domain of p, therefore they have infinite uncertainty in momentum which leads to infinite energy:

$$\langle\psi_\lambda|\frac{p^2}{2m}|\psi_\lambda\rangle = divergent. \tag{54}$$

Unlike ordinary quantum mechanics, in this formulation there is now a finite limit to the localizability.

HILBERT-PÓLYA CONJECTURE

When investigating the non trivial zeroes of the Riemann Zeta function, if one assumes the validity of the Riemann Hypothesis, the nontrivial zeros appear to be closely related to the spectrum of a self-adjoint operator. The *Hilbert-Pólya conjecture* proposes that the imaginary parts of the nontrivial zeros of $\zeta(z)$ correspond to the spectrum of a "Hamiltonian" operator. To clarify, the operator in question would not have units of energy and therefore would only play the role of a Hamiltonian, which is also defined to be Hermitian and self-adjoint. Observations into this connection have found that the spacing of the zeros of the zeta function on the line $\mathbb{R}(z) = \frac{1}{2}$ and the spacing of the eigenvalues of a Gaussian unitary ensemble of Hermitian random matrices have the same distribution [44]. Berry and Keating proposed a classical counterpart of such a "Hamiltonian' with the form $\hat{H} = xp + px$.

Bender-Brody-Müller Hamiltionian

Quantizing gravity is a current problem in modern physics that has no widely agreed on approach. General arguments suggest that regardless of the final form, quantum gravity theory should involve a non-zero minimal distance scale Δx_0. String theoretic arguments lead to such a minimum absolute length scale (see [31] and the references therein for a survey). Many works [1, 2, 3, 7, 4, 5, 6] have shown how a modification of the standard quantum commutator, $[\hat{x}, \hat{p}] = i$, leads to a minimal length scale (we choose units such that $\hbar = 1$). Modifying the standard spatial and/or momentum operators, $[\hat{x}_i, \hat{x}_j]$ and $[\hat{p}_i, \hat{p}_j]$ to be non-zero has been shown in [26]. The two approaches of modifying either $[\hat{x}, \hat{p}]$ or $[\hat{x}_i, \hat{x}_j]$ and/or $[\hat{p}_i, \hat{p}_j]$ are related. Hossenfelder presents a current overview of minimal length scales arising from quantum gravity [27]. In this work, the focus will be on the introduction of a minimal length scale via a modification of $[\hat{x}, \hat{p}]$.

A simple modification of the position operator ($\hat{x}$) and momentum operator ($\hat{p}$) from [7], leads to a commutation relationship of the form

$$[\hat{x}, \hat{p}] = i(1 + \beta \hat{p}^2) \ . \tag{55}$$

It is assumed that β is an arbitrary parameter of quantum gravity. The standard relationship between the quantum commutators and uncertainties gives

$$\Delta x \Delta p \geq \frac{1}{2}\left(1 + \beta \Delta p^2 + \beta \langle \hat{p} \rangle^2\right) \ , \tag{56}$$

which results in a minimal distance of $\Delta x_0 = \sqrt{\beta}$. This approach can be criticized because it is purely phenomenological. The parameter β is not determined, and even the specific form of the modified commutation relationship in (55) is an assumption. The undetermined parameter β is similar to the introduction of the reduced Planck's constant, $\hbar$, which was originally introduced as a parameter to fit the observed black body spectrum.

Various physical motivations exist which support a modified commutator and uncertainty relationship, similar to those presented in equations (55) and (56). In reference [32], arguments are made that at low energy the uncertainty relationship is dominated by the Compton length of an object which leads to the usual relationship $\Delta x \sim \frac{1}{\Delta p}$. At high energy, the uncertainty relationship is dominated by the Schwarzschild radius of the object which leads to relationship $\Delta x \sim \Delta p$. Combining these regimes linearly leads to a modified uncertainty relationship similar to (56). In addition, string theoretic arguments [33, 34, 35] based on looking at colliding strings in the eikonal limit also support the idea of a modified uncertainty relationship. The parameter β is found to be related to the Planck scale in all of these approaches [32, 33, 34, 35].

This work proposes a method to obtain a modified commutation relationship which is considered to be somewhere in between the phenomenological approach of [7] and the physical approach of [32, 33, 34, 35]. This method is motivated by the Bender-Brody-Müller approach [8] to the Riemann hypothesis [36]. There is a clear connection between the distribution of the prime numbers and the non-trivial zeros of the Riemann zeta function. The Riemann hypothesis suggests that all the non-trivial zeros have a real part equal to $\frac{1}{2}$. When calculating the number of physical dimensions in string theory, the Riemann zeta function appears. This is suggestive of a link between the properties of the zeta function and a minimal length scale. The Riemann zeta function is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt \tag{57}$$

where $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$ is the usual gamma function. Using the integral expression in (57) one obtains a reflection formula for the Riemann zeta function

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z). \tag{58}$$

It can be seen in (58) that the Riemann zeta function has *trivial* zeros at the negative even integers, $z = -2n$ due to the $\sin(\pi z/2)$ term. It was initially noticed by Riemann that there exists non-trivial zeros occurring on the line

$Re(z) = \frac{1}{2}$. With specific examples at the complex values $z_n = \frac{1}{2} + it_n$ where $n = 1, 2, 3....$ and $t_1 = 14.135$, $t_2 = 21.022$, $t_3 = 25.011$ *etc.* The Riemann hypothesis states that all of these nontrivial zeros lie on this line $z = \frac{1}{2} + it$.

The imaginary part of the non-trivial zeros of the Riemann zeta function seem to occur in discrete intervals. It has been suggested that the nature of the non-trivial zeros are related to an eigenvalue problem. The Hilbert-Polya conjecture suggests that there exists some operator, $\hat{H}$, whose eigenvalues are exactly the imaginary parts of the non-trivial zeros of the Riemann zeta function. The operator $\hat{H}$ is conventionally called the "Hamiltonian", although it is not connected with the energy of any system, and does not have the dimensions of energy. Under certain conditions quantum theories should reduce to their classical counterparts. Berry [37] and Keating [38] suggest a proposal of a quantum version of the operator $\hat{H}$ that reduces to the classical operator $H = xp$. Their proposed "Hamiltonian" is $\hat{H} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$ in which the order of the quantum operators $\hat{x}$ and $\hat{p}$ matters. This form of $\hat{H}$ is proportional to the one dimensional virial operator which is the generator for scaling/dilation transformations. The appearance of a scaling/dilation transformation is consistent in a conformal theory with no length scale, these transformations represent symmetries of such a theory. This work argues for a theory which breaks scaling/dilation symmetry, introducing a minimal length scale. In looking for a theory with a minimal, absolute length scale, the appearance of a modified or broken dilation symmetry makes sense. Below we introduce an operator $\hat{\Delta}$ which leads to a minimum length and modifies/breaks the virial operator and the related dilation symmetry. In the discussion below, the nomenclature of

"Hamiltonian" operator is dropped and is simply referred to as either operator or modified virial operator.

This work proposes a modification of $\hat{x}$ and $\hat{p}$ such that they align with the recent attempt of Bender-Brody-Müller to address the Riemann hypothesis through the Berry-Keating program. The modified operator proposed by Bender-Brody-Müller [8] is

$$\hat{H} = \hat{\Delta}_{BBM}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}_{BBM} \ , \tag{59}$$

where $\hat{\Delta}_{BBM} = 1 - e^{-i\hat{p}\Delta x}$.

The exponential in $\hat{\Delta}_{BBM}$ is well known as being a shift operator. When $\hat{\Delta}_{BBM}$ acts on an analytic function $f(x)$, one can see that it is a difference operator, $\hat{\Delta}_{BBM}f(x) = f(x) - f(x - \Delta x)$ between the values of the function at x and $x - \Delta x$. In the original work of Bender-Brody-Müller, Δx equals 1 so that the operator was the unit difference operator. For this analysis, the explicit distance scale Δx is retained so that the modifications of the commutator that will lead to a minimal distance can be discussed. The modified operator in (59) is a combination of the virial operator ($i.e.$ $(\hat{x}\hat{p} + \hat{p}\hat{x})$) and the discrete difference operator $\hat{\Delta}_{BBM} = 1 - e^{-i\hat{p}\Delta x}$ and its inverse. The way in which (59) would break conformal symmetry and lead to such an absolute minimal length scale is unclear. Furthermore, applying the operator in (59), one can see that it *is not* successful in introducing a minimal distance scale. The failure of this operator leads to a modified operator of the following form

$$\hat{\Delta}_{ABS} = \frac{1}{2}(e^{kp} + e^{-kp}) = \cosh(kp) \tag{60}$$

where k could be a real, imaginary, or complex constant [39]. By considering a modified virial operator similar to that in (59) but with $\hat{\Delta}_{BBM}$ replaced by $\hat{\Delta}_{ABS}$ one can see that *it does* result in a modified dilation symmetry and an introduction of an absolute minimal length scale. If there exists a minimum length scale, functions should be approximately constant along intervals smaller than the minimum length scale. The operator $\hat{\Delta}_{ABS}$ takes the average of some function $f(x)$ with respect to some general shift of $\pm ik$,

$$\hat{\Delta}_{ABS}f(x) = \frac{1}{2}(f(x - ik) + f(x + ik)). \tag{61}$$

If one takes $k = \pm i\frac{\Delta x}{2}$ then this equals

$$\frac{1}{2}(f\left(x - \frac{\Delta x}{2}\right) + f\left(x + \frac{\Delta x}{2}\right)), \tag{62}$$

which is an averaging of $f(x)$ over an interval Δx. On intervals that oscillate at length scales less than the minimum length Δx, this operator sends functions approximately to zero. In order to get a modified commutator that leads to a minimal length, as well as modified position and momentum operators that have good limiting behavior, we need the k in (60) to be purely real. For a real k this implies that the $\hat{\Delta}_{ABS}$ gives a shift of the function in the imaginary direction namely

$$\hat{\Delta}_{ABS}f(x) = \frac{1}{2}(f(x - ik) + f(x + ik)). \tag{63}$$

The operator $\hat{\Delta}_{BBM}$ uses a pure imaginary k which results in a shift in the real direction.

Taking into account that $\hat{x}$ and $\hat{p}$ satisfy the standard commutator relation $[\hat{x}, \hat{p}] = i$, from equation (59) the operator can be "pushed" through to the left and annihilated with its inverse operator $\hat{\Delta}^{-1}_{BBM}$. The operator in (59) becomes

$$\hat{H} = (\hat{x}\hat{p} + \hat{p}\hat{x}) - \frac{2\hat{p}\Delta x e^{-i\hat{p}\Delta x}}{(1 - e^{-i\hat{p}\Delta x})} \ . \tag{64}$$

From this modified virial operator, a family of modified position and momentum operators and their modified commutators can be considered. The modified position and momentum operators provided below, lead to a minimum length scale similar to [7]. These modified position and momentum operators are symmetric in an inner-product space which requires the wave function to exponentially decay in the large momentum limit. The modified operator, found below, provides a similar approach to the Riemann hypothesis as suggested by Bender-Brody-Müller [8]. There are open questions [40] and discussions [41] regarding the Bender-Brody-Mueller approach to the Riemann hypothesis. It is not the intention of this work to resolve the major criticism of "What is the Hilbert space used in the construction in reference [8]?". The family of operators presented may provide alternative avenues to resolving this criticism. However, the main goal here is to use operators, such as given in equation (64), to give a top-down motivation for a modified commutation relationship between position and momentum.

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) \tag{65}$$

Modified Position and Momentum Commutator

We assume the modified position and momentum operators in the form

$$\hat{x}' = i(1 + g(p))\partial_p \quad ; \quad \hat{p}' = p(1 + f(p)) \ . \tag{66}$$

The expressions in (66) represent a general way of modifying the position and momentum operators. The form in (66) includes the modified position and momentum operators from reference [7] if one takes $f(p) = 0$ and $g(p) = 1 + \beta p^2$. The form of the modified position and momentum operators in (66) also covers the case of κ-deformed Poincaré algebra from reference [42]. In fact we will find that the form of the $g(p)$ that we obtain in the end involves hyperbolic functions which gives a modified position operator similar in form to the modified Newton-Wigner position operator suggested in [42]. Even more recently, reference [43] gave modified position and momentum operators of the form in equation (66) in order to formulate a relativistic generalized uncertainty principle. To obtain a specific form for the generalized position and momentum operators in (66), we require that the new viral operator, $\hat{x}'\hat{p}' + \hat{p}'\hat{x}'$, with $\hat{x}'$ and $\hat{p}'$ from (66), leads to a modified virial operator like that in (64). This requirement will lead to specific functions, $g(p)$ and $f(p)$, which in turn will give a specific form for the modified position and momentum operators. We are working in momentum space since the extra term in $\hat{H}$ from (64) involves only the momentum operator. Using the operators from (66) we find that the new operator becomes

$$\hat{H} = (\hat{x}'\hat{p}' + \hat{p}'\hat{x}') \tag{67}$$

$$\hat{H} = 2ip(1+f(p))(1+g(p))[\partial_p + i] + i(1+g(p))(f(p)+pf'(p)) + ig(p). \quad (68)$$

The first term in (67) (*i.e.* $2ip(1+g(p))(1+f(p))\partial_p + i$) should correspond to the first term in (64) (*i.e.* $(\hat{x}\hat{p}+\hat{p}\hat{x}) = 2ip\partial_p + i$, using $\hat{x} = i\partial_p$ and $\hat{p} = p$). This correspondence is accomplished by requiring $(1+f(p))(1+g(p)) = 1$, *i.e.*

$$g(p) = \frac{-f(p)}{1+f(p)} \ . \quad (69)$$

With this $g(p)$ the remaining terms in (67) become

$$i(1+g(p))(f(p)+pf'(p)) + ig(p) = \frac{ipf'(p)}{1+f(p)} \ .$$

$f(p)$ is determined by requiring the above expression equal the last term in (64) yielding

$$\frac{ipf'(p)}{1+f(p)} = -\frac{2p\Delta x e^{-ip\Delta x}}{1-e^{-ip\Delta x}} \to \frac{d}{dp}\left[\ln(1+f(p))\right] = \frac{2i\Delta x e^{-ip\Delta x}}{1-e^{-ip\Delta}} \ . \quad (70)$$

Equation (70) is straight forward to solve and yields the solution

$$1+f(p) = C(1-e^{-ip\Delta x})^2 \ . \quad (71)$$

from which it follows

$$1+g(p) = \frac{1}{C(1-e^{-ip\Delta x})^2} \ . \quad (72)$$

Using the modified position and momentum operators from equations (66), (71) and (72), it can be seen that the associated modified commutator becomes

$$[\hat{x}', \hat{p}'] = i\left(1 + \frac{pf'(p)}{1+f(p)}\right) = i + \frac{2p\Delta x}{1 - e^{ip\Delta x}} \ , \tag{73}$$

where the expression for $g(p)$ has been used from (69) to obtain both the intermediate form and the final form used in (71). The first term i is the standard commutator and the second term $\frac{2p\Delta x}{1-e^{ip\Delta x}}$ is the modification coming from the deformation of the position and momentum operators. It is this second term which represents the change associated with a modification of short distance/large momentum behavior coming from quantum gravity. Equation (73) is the modification of the quantum commutator implied by the requirement that the modified position and momentum operators from equations (66), (71), and (72) give the modified virial operator in (59) or (64).

It is a physical requirement that, in the low momentum limit, one should recover the standard operators, *i.e.* $g(p), f(p) \to 0$ as $p \to 0$. It can easily be seen from equations (71) and (72) that $f(p) \to -1$ and $g(p)$ diverges as $p \to 0$. Additionally, as $p \to 0$, we want the commutator in equation (73) to reduce to $[\hat{x}, \hat{p}] = i$. As $(p\Delta x) \to 0$ we see that $\frac{2p\Delta x}{1-e^{ip\Delta x}} \to 2i$, and thus in this limit the commutator in (73) becomes $[\hat{x}', \hat{p}'] \to 3i$ yielding an nonphysical result. As mentioned in the previous section, the operator from (59), which is both a dilation and difference operator $(1 - e^{-ip\Delta x})$, does not modify the commutator in such a way that culminates in a minimal length scale. To remedy this failure, a modification to the operator in (59) or (64) can be made to result in a commutator with the correct physical limit as $p \to 0$, while still preserving the potential approach to the Riemann hypothesis proposed in [8].

We must begin finding a new $\hat{\Delta}$ which will exhibit a more physical behavior in the $p \to 0$ limit, which differs from $\hat{\Delta}_{BBM}$ in (59) and (64). It must also be true that this new $\hat{\Delta}$ still satisfies the approach to the Riemann Hypothesis given in [8]. It is important to note that in the original construction $\hat{\Delta}_{BBM} = 1 - e^{-ip\Delta x} \to 0$ as $(p\Delta x) \to 0$ is a problem. We can avoid this by taking a + sign such that $\hat{\Delta} = 1 + e^{-ip\Delta x}$. This new operator is a kind of averaging transformation of a function between points x and $x - \Delta x$ rather than a difference operator, as is the case with $\hat{\Delta}_{BBM}$. Applying $1 + e^{-ip\Delta x}$ to a function, $f(x)$, one finds $\hat{\Delta} f(x) = f(x) + f(x - \Delta x)$. To make this a true averaging between the points x and $x - \Delta x$, we should divide by $\frac{1}{2}$.

As mentioned previously, a generalized averaging operator $\hat{\Delta}_{ABS} = \frac{1}{2}(e^{kp} + e^{-kp}) = \cosh(kp)$ will be considered as given in (60) which is symmetric in p. Unlike the Bender-Brody-Müller case, here we find that we need to have k be real rather than imaginary ($i.e.$ $k = \pm i\Delta x$). With $\hat{\Delta}_{ABS} = \cosh(kp)$, the modified virial operator becomes

$$\begin{aligned} \hat{H} &= \hat{\Delta}^{-1}_{ABS}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}_{ABS} = \hat{x}\hat{p} + \hat{p}\hat{x} + 2p(\hat{\Delta}^{-1}_{ABS}[x, \hat{\Delta}_{ABS}]) \\ &= \hat{x}\hat{p} + \hat{p}\hat{x} + \frac{2ip}{\cosh(kp)}\partial_p(\cosh(kp)) \\ &= \hat{x}\hat{p} + \hat{p}\hat{x} + 2ipk\tanh(kp)\ . \end{aligned} \tag{74}$$

One obtains a differential equation, similar to (70), which reads $\frac{ipf'(p)}{1+f(p)} = 2ipk\tanh(kp)$. This gives the following solution for $f(p)$:

$$1 + f(p) = C\cosh^2(kp) \tag{75}$$

which can be used in (69) to obtain $g(p)$

$$1 + g(p) = C^{-1}\text{sech}^2(kp). \tag{76}$$

The functions $f(p), g(p)$ from (75) and (76) inserted in (66) give the modified position and momentum operators

$$\hat{x}' = i\text{sech}^2(kp)\partial_p \quad ; \quad \hat{p}' = \cosh^2(kp)p \tag{77}$$

where $C = 1$ so $\hat{p}' \to p$ and $\hat{x}' \to i\partial_p$ as $p \to 0$. These operators are symmetric with respect to the inner product $\langle\psi(p)|\phi(p)\rangle = \int_{-\infty}^{\infty}\cosh^2(kp)\overline{\psi(p)}\phi(p)dp$. This inner product leads to the norm $\|\psi\|^2 = \int_{-\infty}^{\infty}\cosh^2(kp)|\psi(p)|^2dp$. One needs exponential suppression of the wave function at high momentum to counter the $\cosh^2(kp)$ factor such that the norm will be finite and will provide normalizeable states. The modified commutation relations become

$$[\hat{x}', \hat{p}'] = i\text{sech}^2(kp)\partial_p[\cosh^2(kp)p] \tag{78}$$

$$[\hat{x}', \hat{p}'] = i\left(1 + 2kp\tanh(kp)\right) \tag{79}$$

If $kp \ll 1$, the right hand side of (78) can be expanded using $\tanh(kp) \approx kp + \mathcal{O}(kp)^3$ with the result

$$[\hat{x}', \hat{p}'] \approx i(1 + 2k^2p^2). \tag{80}$$

This $kp \ll 1$ limit gives a commutator which is the same as the phenomenological commutator given by (55) with $\beta = 2k^2$, provided that k is real so that $k^2 > 0$. This is required since one needs $\beta > 0$ in (55) in order to get a minimal length. For the operator $\hat{\Delta}_{ABS} = \cosh(kp)$, in order to recover a minimal length in a method similar to that in reference [7], we require that k be real. In contrast for the Bender-Brody-Müller operator, $\hat{\Delta}_{BBM} = 1 - e^{-ip\Delta x}$, one has $k = -i\Delta x$ which gives $k^2 < 0$ and no minimal length.Along with the results for the $p\Delta x \to 0$ limit of the modified commutator from (73), this shows that the Bender-Brody-Müller operator and Hamiltonian do not lead to a minimal length scale. This appearance of a minimal length scale, for real k (*i.e.* $k^2 > 0$ and $\beta > 0$), can be linked to the form of the modified virial operator in (74) which breaks the dilation symmetry associated with the virial operator $\hat{x}\hat{p} + \hat{p}\hat{x}$, via the scale dependent averaging operator $\hat{\Delta}_{ABS} = \cosh(kp)$.

It can be seen from the calculation leading to (78) that by modifying the operator $\hat{\Delta}_{BBM}$ from $1 - e^{-ip\Delta x}$ (the form taken in [8]) to $\cosh(kp)$ a physically reasonable modification of the position and momentum commutator is yielded. An intention of this work was to connect the modification of the position and momentum commutator to the Bender-Brody-Müller variant of the Berry-Keating program. By choosing $\hat{\Delta}_{ABS} = \cosh(kp)$, one is still allowed to follow a similar construction to the operator proposed in [8] to address the Riemann Hypothesis. We will find that there are several variants of $\hat{\Delta}$ which work.

The basic idea of the Berry-Keating program is that there exists some operator, $\hat{H}$, which satisfies an eigenvalue equation $\hat{H}\Psi = E\Psi$ whose eigenvalues, E, give the imaginary part of the non-trivial zeros of the Riemann zeta function. The operator we consider is of the Bender-Brody-Müller form $\hat{H} = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}$ where we take $\hat{\Delta}_{ABS} = \cosh(kp)$. Following [8] we begin by re-writing the eigenvalue equation as

$$\hat{H}\Psi = E\Psi \rightarrow (\hat{x}\hat{p} + \hat{p}\hat{x})(\hat{\Delta}_{ABS}\Psi) = E(\hat{\Delta}_{ABS}\Psi) \ , \tag{81}$$

which is an eigenvalue equation for $\hat{\Delta}_{ABS}\Psi$ with respect to the operator $\hat{x}\hat{p} + \hat{p}\hat{x}$. Using the standard coordinate space representation of the position and momentum operators, $\hat{x} = x$ and $\hat{p} = -i\partial_x$, the eigenfunctions and eigenvalues to (81) are $\hat{\Delta}_{ABS}\Psi(z, x) = Ax^{-z}$ and $E_z = i(2z - 1)$ respectively, where A is a constant.

There is a Ψ such that $\hat{\Delta}_{ABS}\Psi(z, x) = Ax^{-z}$. By applying e^{kp} to an analytic function, $f(x)$ the result is $e^{kp}f(x) = f(x - ik)$ *i.e.* e^{kp} which happens to be a generalized shift operator, $f(x)$ by ik. When $k = \pm i$ this is a shift of $x \rightarrow x \pm 1$. It follows

$$\cosh(kp)f(x) = \frac{1}{2}(e^{kp} + e^{-kp})f(x) = \frac{1}{2}[f(x - ik) + f(x + ik)].$$

In reference [8] where $\hat{\Delta} = 1 - e^{-ip}$, the solution to $\hat{\Delta}\Psi(z,x) = Ax^{-z}$ was the Hurwitz zeta function, $\zeta(z, x+1)$, defined as

$$\Psi(z,x) \propto \zeta(z,x+1) = \sum_{n=0}^{\infty} \frac{1}{(n+x+1)^z} \ . \tag{82}$$

By imposing the boundary condition $\Psi(z,0) = 0$, this 'forces' the Riemann zeta function $\zeta(z,1)$ to equal 0. If the spectrum of this operator can be shown to be real, then the z's have the form $\frac{1}{2} + it$ for real t, *i.e.* the non-trivial zeros of the zeta function are on this critical line. The goal of this research is not to prove the Riemann hypothesis and therefore no argument is provided on the reality of the spectrum.

For the case when $\hat{\Delta}_{ABS} = \cosh(kp) = \frac{1}{2}(e^{kp} + e^{-kp})$, we use the Hurwitz-Euler eta function [44]

$$\eta(z,x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x+1)^z} \ , \tag{83}$$

and show that it solves the equation $\hat{\Delta}_{ABS}\Psi(z,x) = Ax^{-z}$. This function is an alternating sign version of the Hurwitz zeta function of (82). For $x = 0$, $\eta(z,x+1)$ becomes the well known Dirichlet eta function [22]

$$\eta(z,1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^z} \ , \tag{84}$$

an alternating sign version of the Riemann zeta function. The Hurwitz-Euler eta function satisfies $\hat{\Delta}_{ABS}\Psi(z,x) = Ax^{-z}$ by applying the shift $x \to \frac{x}{2ik} - \frac{1}{2}$ in (83) and arrive at

$$\eta\left(z, \frac{x}{2ik} + \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{x}{2ik} + \frac{1}{2})^z} \ . \tag{85}$$

Recalling that $\hat{\Delta}_{ABS} f(x) = \frac{1}{2}(f(x - ik) + f(x + ik))$ and applying this to $\eta\left(z, \frac{x}{2ik} + \frac{1}{2}\right)$ yields

$$\begin{aligned}
\hat{\Delta}_{ABS}\left[\eta\left(z, \frac{x}{2ik} + \frac{1}{2}\right)\right] &= \frac{1}{2}\left[\eta\left(z, \frac{x}{2ik}\right) + \eta\left(z, \frac{x}{2ik} + 1\right)\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{x}{2ik})^z} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{x}{2ik} + 1)^z} \\
&= \frac{1}{2}\left(\frac{x}{2ik}\right)^{-z} \propto x^{-z} \ ,
\end{aligned} \tag{86}$$

where the alternating sign of the two series makes all the terms cancel between the series except for the $n = 0$ term of the first series. This shows that $\eta\left(z, \frac{x}{2ik} + \frac{1}{2}\right)$ satisfies the equation $\hat{\Delta}_{ABS}\Psi(z, x) = Ax^{-z}$.

By imposing the boundary condition that the functions should be equal to zero at $x = ik$, *i.e.* $\eta\left(z, \frac{ik}{2ik} + \frac{1}{2}\right) = \eta(z, 1) = 0$, the Bender-Brody-Müller approach is continued and makes the Dirichlet eta function equal to zero. The Dirichlet eta function has the same non-trivial zeros as the Riemann zeta function. This can be seen through the functional relationship between the Riemann zeta function and Dirichlet eta function [22]:

$$\eta(z, 1) = (1 - 2^{1-z})\zeta(z, 1).$$

Therefore, the trivial and non-trivial zeros of the Riemann zeta function are also zeros of the Dirichlet eta function. The Dirichlet eta function also has additional trivial zeros of the form $z = 1 + 2\pi ik/\ln(2)$ with $k \in \mathbb{Z}$ such that the pre-factor

$(1 - 2^{1-z}) = 0$. It is argued in [8] that the trivial zeros at $z = -2n$ of $\zeta(z, 1)$ correspond to the eigenfunctions, which diverge as $x \to \infty$. If this argument holds, then those eigenfunctions not belong to the function space. The additional trivial zeros for the Dirichlet eta function at $z = 1 + 2\pi ik/\ln(2)$ could also be discarded for similar function space reasons. Without a well-defined function space, these arguments are suggestive at best. It is the boundary conditions that link the non-trivial zeros of the Dirichlet eta function and the solutions of the eigenvalue equation. These zeros are exactly the non-trivial zeros of the Riemann zeta function and this shows that the approach in [8] to addressing the Riemann Hypothesis also works for $\hat{\Delta}_{ABS} = \cosh(kp)$.

The present construction differs from that of Bender-Brody-Müller (aside from the use of Hurwitz-Euler eta and Dirichlet eta functions versus Hurwitz zeta and Riemann zeta functions) in that now the non-trivial zeros are determined by setting the function equal to zero at $x = ik$ as opposed to $x = 0$ as in [8]. The motivation of this shift of the location of the zeros of the eigenfunction, from $x = 0$ to $x = ik$, can be seen by considering $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$. If one follows the steps in equations (81) - (86) it can be shown that $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$ also works for a construction similar to that given by Bender-Brody-Müller. The Ψ satisfying $\hat{\Delta}\Psi(z, x) = Ax^{-z}$ is now of the form $\Psi(z, x) = \eta(z, \frac{x}{2ik})$ (note the lack of $+\frac{1}{2}$). Therefore, when $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$, the boundary condition is set at $x = 0$, *i.e.* $\eta(z, x = 0) = 0$. Finally, we can get from $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$ to $\hat{\Delta}_{ABS} = \cosh(kp) = \frac{1}{2}(e^{kp} + e^{-kp})$ by applying e^{-kp} to $\frac{1}{2}(1 + e^{2kp})$. The operator e^{-kp} shifts functions by ik (*i.e.* $e^{-kp}f(x) \to f(x + ik)$) which would shift the boundary condition from $x = 0$, for the $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$ case, to $x = ik$, for the $\hat{\Delta}_{ABS} = \cosh(kp) = \frac{1}{2}(e^{kp} + e^{-kp})$ case.

CONCLUSIONS AND SUMMARY

This work began with an investigation to prime number theory and the mathematical framework of the Riemann zeta function, starting with the Dirichlet series. String theoretic arguments for the number of physical dimensions, which invoked the Riemann zeta function, were a major connection point between the zeta function and space-time structure. Analytic continuation techniques were applied to the zeta function to find the complete set of trivial zeros. Assuming the validity of the Riemann Hypothesis, the imaginary part of the non-trivial zeros of the zeta function were assumed to correspond to a "Hamiltonian" operator via the Hilbert-Pólya conjecture. From the Berry-Keating program, a starting point was provided in finding an operator that is directly related to the non-trivial zeros the zeta function. From Kempf, Mangano, and Mann a modification to the standard quantum mechanical operators was introduced which lead to a minimum length scale. The Bender-Brody-Müller variant of the Berry-Keating program provided a guide to give a specific form for a modified commutator. The main result of this work is that we arrive at a modification of the standard quantum position and momentum commutation relationship. These modified operators and commutators are given in equations (78) and (79). This differs from earlier proposals for modified operators and commutators, such as (56), which are phenomenologically motivated. The modified operators and commutators lead to a minimum length scale and a modified dilation symmetry, generated by $\hat{\Delta}^{-1}_{ABS}(\hat{x}\hat{p}+\hat{p}\hat{x})\hat{\Delta}_{ABS}$. In addition, several different variants of $\hat{\Delta}$, used in defining the operator $\hat{H}=\hat{\Delta}^{-1}(\hat{x}\hat{p}+\hat{p}\hat{x})\hat{\Delta}$, were found that allow one to tackle the

Riemann hypothesis in the way proposed in reference [8]. In addition to $\hat{\Delta}_{BBM} = 1 - e^{-i\hat{p}\Delta x}$, used by Bender-Brody-Müller, we have found that $\hat{\Delta}_{ABS} = \frac{1}{2}(e^{kp} + e^{-kp})$ and $\hat{\Delta} = \frac{1}{2}(1 + e^{kp})$ also lead to similar approaches to the Riemann hypothesis.

This work concludes with the remark that the analysis here can be used to show that modifications of the quantum commutator, such as given in (55), can be connected with different modifications of the position and momentum operators. In [7] the modified position and momentum operators connected with $[\hat{x}, \hat{p}] = i(1 + \beta\hat{p}^2)$ were given as

$$\hat{x} = i(1 + \beta p^2)\partial_p \quad ; \quad \hat{p} = p \ . \tag{87}$$

The position operator is changed but the momentum operator is not. Using the analysis of position and momentum operators starting with (66), but having in mind the modified commutator given in (55), we find that the ansatz functions are $1 + f(p) = e^{\beta p^2/2}$ and $1 + g(p) = e^{-\beta p^2/2}$. These lead to modified position and momentum operators of the form

$$\hat{x}' = ie^{-\beta p^2/2}\partial_p \quad ; \quad \hat{p}' = e^{\beta p^2/2} p \ . \tag{88}$$

Both sets of modified operators, those from equation (87) and equation (88), lead to the same modified commutation relationship (55).

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