ABSTRACT

THE WELL-COVERED DIMENSION OF THE ADJACENCY GRAPH
OF GENERALIZED QUADRANGLES

Generalized quadrangles are a type of geometric incidence structure of points and lines. In this work, we investigate the well-covered dimension, which is a parameter from the field of graph theory, of graphs related to generalized quadrangles. In order to calculate the well-covered dimension, we must first associate a graph to the geometric structure, called its adjacency graph, which is obtained from the structure in a specific way.

The well-covered dimension of a graph is calculated using sets of vertices, called maximal independent sets, in which no two vertices of the set are adjacent and the set contains a maximal number of vertices. We will show that the well-covered dimension of the adjacency graph of a generalized quadrangle with all regular points is zero by finding maximal independent sets of different cardinalities.

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THE WELL-COVERED DIMENSION OF THE ADJACENCY GRAPH
OF GENERALIZED QUADRANGLES

by

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INTRODUCTION

There are fields within mathematics which, to an observer, may seem vastly unrelated at first. However, as one looks closer, connections between all of these different areas begin to emerge. The motivation for this work is a desire to study the graphs associated to geometric objects, using tools from linear algebra and combinatorics. By combining all of these different ideas, we will be able to discover properties about these geometric structures that have implications in many different fields.

Graphs and Incidence Structures

We begin with the introduction of a few ideas from graph theory.

Definition 1. A graph $G$ consists of a finite set $V(G)$ of vertices and a set $E(G)$ of edges, where the edges of the graph are subsets of $V(G)$ consisting of two distinct vertices.

Edges are thought of as segments connecting two vertices. We will only be considering graphs that do not contain ‘loops’ (edges with only one endpoint) or multiple edges between any two given vertices.

Example 2. The following graph has vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}$. 
Definition 3. The number of edges incident to a given vertex is called the degree of that vertex.

In Example 2, the degree of \( v_1 \) is 1, the degree of \( v_2 \) is 3, the degree of \( v_3 \) is 2, and the degree of \( v_4 \) is 2.

Definition 4. If each vertex of a graph has the same degree, we say the graph is regular.

Therefore, the figure in Example 2 is not a regular graph.

Definition 5. Two vertices of a graph are adjacent if there is an edge connecting them.

In Example 2, \( v_1 \) and \( v_2 \) are adjacent.

Definition 6. A bipartite graph is a graph with two disjoint sets of vertices such that no two vertices within the same set are adjacent.

Definition 7. An independent set is a set of vertices of a graph in which no two vertices are adjacent. A maximal independent set, abbreviated MIS \(^1\), is

\(^1\)The abbreviation MIS will be used both for singular and plural instances
an independent set that is not a subset of any other independent set. That is, adding any one vertex to a maximal independent set will result in the set no longer being independent. A graph is said to be well-covered if every maximal independent set has the same cardinality.

**Example 8.** The following graph has vertex set

\[ V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}. \]

![Figure 2. A MIS in a small graph](image)

The vertices shown in red represent one MIS, comprised of vertices \( \{v_1, v_3, v_6\} \).

We can find other maximal independent sets by starting with a set containing a single vertex, and completing to a maximal independent set, one vertex at a time. This completion is not necessarily unique, as we will demonstrate.

Start with \( v_7 \). Since \( v_7 \) is adjacent to \( v_1, v_2, \) and \( v_3 \), none of those can be in an independent set with \( v_7 \). So we choose to add to our independent set \( v_4 \). \( v_4 \) is similarly adjacent to \( v_1 \) and \( v_3 \), but also \( v_6 \). So we cannot add any of
those vertices to the independent set. However, $v_5$ is not adjacent to either $v_7$ or $v_2$, so we can add it to the independent set. The only remaining available vertex would be $v_6$, but $v_6$ is adjacent to $v_5$, so we cannot add it to the independent set. Since there remain no more vertices that could be added to the independent set, $\{v_7, v_4, v_5\}$ is a maximal independent set.

Had we chosen $v_6$ to be added first to the independent set $\{v_7\}$, there would not have been any other vertices to add to the set $\{v_7, v_6\}$ and preserve its independence. Thus $\{v_6, v_7\}$ is a maximal independent set as well. It follows that this graph is not well-covered.

Incidence is an abstract idea of a relation between two sets of objects. In our work, we consider finite sets of points and lines in geometric objects, and we will refer to the incidence relation by saying that a point is on a line, or that a line contains a point.

**Definition 9.** A finite incidence structure $(P, L, I)$ consists of a set of points $P$, a set of lines $L$, where $I$ denotes the incidence of a point on a line.

**Example 10.** The following picture consists of a set of points $P = \{v_1, v_2, v_3\}$ and a set of lines $L = \{\ell, m\}$. The incidence relation $I$ relates some of the points and lines to each other. The points $v_1$ and $v_2$ are incident to the line $\ell$, and the points $v_2$ and $v_3$ are incident to the line $m$, because $v_1$ and $v_2$ lie on the line $\ell$, and $v_2$ and $v_3$ lie on the line $m$. Similarly, the line $\ell$ is incident to the points $v_1$ and $v_2$. 
We will be using graphs to study certain incidence structures called generalized quadrangles, which will be introduced in the next section (see Definition 27).

There are a few different ways to associate a graph to a geometric object or incidence structure. The following definitions describe two such associated graphs.

**Definition 11.** The Levi graph associated to a configuration or incidence structure is the bipartite graph obtained by defining one set of vertices, $P$, corresponding to the points in the configuration and a second set of vertices, $L$, corresponding to the lines in the configuration. An edge connects two vertices $p$ and $l$ if and only if the point is incident with the line in the configuration.

The Levi graphs of some well-known configurations and geometric structures have already been studied (see, for example, [8] and [9]). These results will be discussed later.

Another type of graph one can associate to a geometric object is called the adjacency graph. This object will be the focus of our study.
Definition 12. Given an incidence structure $S = (P, L, I)$, we define the adjacency or collinearity graph of $S$, denoted $A(S)$, as the graph consisting of vertices given by the points in $S$, and edges connecting pairs of points if and only if they are collinear within the incidence structure (i.e., are incident to the same line).

The following example will illustrate an incidence structure, its Levi graph, and its collinearity graph.

Example 13. Figure 4 illustrates the Fano plane, a well-known incidence structure. It is also the smallest projective plane.

![Figure 4. The Fano plane](image)

Figure 5 shows the Levi graph of the Fano plane. The vertices numbered 1-7 correspond to the points of the Fano plane, as in Figure 4. The remaining vertices correspond to the lines of the Fano plane, labeled $\ell_1 - \ell_7$, as in Figure 4.
Figure 5. The Levi graph of the Fano plane

Figure 6 shows the adjacency graph of the Fano plane. The vertices of the adjacency graph, labeled 1 – 7, correspond to the points of the Fano plane.

The Levi graphs of many geometric configurations, including generalized quadrangles, are well understood (see [8]). This thesis is concerned with investigating properties of adjacency graphs.
One method of analyzing a graph (and therefore the incidence structure it is associated to), involves the study of algebraic structures.

We begin by introducing a few objects related to graphs that are necessary for our study.

**Definition 14.** A field is a set $\mathbb{F}$ with two binary operations, usually denoted $+$ and $\cdot$, such that $\mathbb{F}$ is a commutative group under $+$ with identity usually denoted by 0, $\mathbb{F} \setminus \{0\}$ is a commutative group under $\cdot$ with identity usually denoted by 1, and distributive properties hold. The characteristic of a field $\mathbb{F}$ is the smallest positive integer $p$ such that $px = 0$ for all $x \in \mathbb{F}$, or 0 if no such $p$ exists (see [1]).

**Example 15.** The characteristic of the field of rational or real numbers is 0; the field $\mathbb{Z}_p$ has characteristic $p$.

It is well-known that the characteristic of a field must be either zero or a prime number $p$.

**Definition 16.** A vector space $(\mathbb{V}, +)$ over a field $\mathbb{F}$ is an abelian group, that also has the following properties. For all vectors $x, y, z \in \mathbb{V}$ and all scalars $a, b \in \mathbb{F}$:

- $a(bx) = (ab)x$ (Associativity of scalar multiplication)
- $(a + b)x = ax + bx$ (Distributivity of scalar sums)
- $a(x + y) = ax + ay$ (Distributivity of vector sums)
- $1 \cdot x = x$ (Scalar multiplication identity)
If a set \( S = \{v_1, v_2, \ldots \} \) is a subset of a vector space \( V \), and if every vector in \( V \) can be written as a linear combination of vectors in \( S \), then \( S \) spans \( V \). A basis of a vector space is a set of linearly independent vectors which span the entire vector space. Any two bases of a vector space have the same cardinality. The dimension of a vector space \( V \) is defined to be the cardinality of any of its bases.

This concept of the dimension of a vector space will be extended in the next section, where a similar idea of dimension, relating to a graph, will be introduced.

**The Well-Covered Dimension of a Graph**

Recall that a well-covered graph is defined as one in which all maximal independent sets have the same cardinality (cf Definition 7). This notion was first presented in 1970 by Plummer (see [10]). Well-covered graphs have been studied in many papers over the past few decades (see [8], [2]). In 1999, Caro and Yuster defined the well-covered space of a graph, and first introduced the idea of the dimension of this vector space (see [6]). Brown and Nowakowski later introduced new ideas and notation in this area in [4], and they presented results about the well-covered dimension of many families of graphs.

The following definitions follow Brown and Nowakowski (see [4]).

**Definition 17.** Given a fixed field \( F \), and a graph \( G \), a weighting is a function \( f : V(G) \to F \) that assigns a value from the field \( F \) to each vertex of \( G \).

**Definition 18.** A well-covered weighting is a weighting \( f \) such that \( \sum_{x \in M} f(x) \) is constant for every maximal independent set \( M \) of \( G \).
Brown and Nowakowski noted in [4] that well-covered graphs are precisely those graphs for which the function $1_G : V(G) \rightarrow \mathbb{F}$, which maps $v \mapsto 1$ for all vertices $v$, is a well-covered weighting (over any field of characteristic 0).

**Definition 19.** The set of well-covered weightings of a graph $G$ over a field $\mathbb{F}$ is a vector space, called the well-covered space of $G$. The dimension of this vector space is called the well-covered dimension of the graph, denoted $wcdim(G, \mathbb{F})$.

Recall that a graph is called well-covered if the cardinality of all its maximal independent sets is a constant (see Definition 7).

**Definition 20.** A graph is anti-well-covered if its well-covered dimension is equal to zero.

**Definition 21.** If $M_1, M_2, \ldots, M_{t+1}$ are the maximal independent sets of $G$, then well-covered weightings are precisely the solutions to the associated linear system

$$\sum_{v \in M_1} x_v = \sum_{v \in M_{t+1}} x_v$$

$$\sum_{v \in M_2} x_v = \sum_{v \in M_{t+1}} x_v$$

$$\vdots$$

$$\sum_{v \in M_t} x_v = \sum_{v \in M_{t+1}} x_v$$

where $x_v$ represents the weight of $v$ under a weighting.
We can write this linear system in matrix form as $A_Gx = 0$. $A_G$ is called the associated matrix for the graph $G$, and then $wcdim(G)$ equals the nullity of $A_G$ (over $\mathbb{F}$), so we have

$$wcdim(G, \mathbb{F}) = |V(G)| - \text{rank}(A_G)$$

It is important to note that the well-covered dimension of a given graph $G$ may depend on the characteristic of the field over which it is calculated. If the well-covered dimension is not affected by the characteristic of the field, we will simply write $wcdim(G)$, as in [4].

In 1998, Caro, Ellingham, and Ramey generalized the study of well-covered graphs. They published results on the well-covered space of a graph. One year later, in Caro and Yuster’s work, (see [6]), the authors show that the well-covered dimension of a tree is equal to the number of vertices of degree one.

In Brown and Nowakowski’s 2005 paper (see [4]), the authors establish an upper bound on the well-covered dimension of a graph of order $n$ using the chromatic number of the graph.

**Theorem 22** (Brown and Nowakowski). Let $G$ be a graph of order $n$. Then $wcdim(G) \leq n - \chi(G) + 1$.

Brown and Nowakowski also have results for the well-covered dimension of the disjoint union and join of two graphs, complements of $k$-trees, chordal graphs, and many others.

More recently, Birnbaum et al. established results for the well-covered dimension of crown graphs, Cartesian products of paths, cycles, and complete
graphs (see [2]). For gear graphs and cycles, the authors found that as long as the graph is sufficiently large (more than 7 vertices), the well-covered dimension of the graph is 0. They also proved a result on the Cartesian products of many types of graphs. In this result, $C_n$ denotes the cycle of length $n$, and $P_n$ denotes the path of length $n$.

**Theorem 23** (Birnbaum et. al. [2]). *Over any field $\mathbb{F}$,

1. $wcdim(C_n \times C_m, \mathbb{F}) = 0$, for $m, n \geq 6$.

2. $wcdim(P_n \times P_m, \mathbb{F}) = 0$, for $m, n \geq 5$.

3. $wcdim(P_n \times C_m, \mathbb{F}) = 0$, for $m \geq 6$ and $n \geq 5$.

In the following examples, we will calculate the well-covered dimension of some simple graphs.

**Example 24.** Consider the graph $G$ in Figure 7. In order to calculate its well-covered dimension, we first need to find all maximal independent sets.

![Figure 7. A simple graph](image)

The MIS of this graph are $\{v_1, v_3\}, \{v_1, v_4\}, \{v_2\}$. We will denote them in the same order $M_1, M_2, M_3$. Now we must form the associated matrix for the graph $G$. 
Then we will take the linear system

\[
\sum_{v \in M_1} x_v = \sum_{v \in M_3} x_v \\
\sum_{v \in M_2} x_v = \sum_{v \in M_3} x_v
\]

which will give us the associated matrix

\[
\begin{pmatrix}
1 & -1 & 1 & 0 \\
1 & -1 & 0 & 1
\end{pmatrix}
\]

Thus \( wcdim(G) = |V(G)| - \text{rank}(A_G) = 4 - 2 = 2 \).

**Example 25.** The well-covered dimension of the graph in Figure 8 depends on the characteristic of the field over which it is calculated. This graph is one of the smallest with this property.

![Figure 8. A graph whose dimension changes with characteristic](image)

As in the previous example, we must first find the maximal independent sets. The MIS are: \( \{v_1, v_3\}, \{v_1, v_6\}, \{v_2, v_4, v_6\}, \{v_2, v_5\}, \{v_3, v_5, v_7\}, \{v_4, v_7\} \).
Using these maximal independent sets, we can find the associated matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}
\]
in its reduced form. Note that there is a 2 in the bottom row. In any field with characteristic 2, 2 is equivalent to 0. In this case, the bottom row of this matrix will be all zero, and this will increase the nullity of the matrix by 1.

Thus, over any field with characteristic 2, the well-covered dimension of this graph is 3, and over any field with characteristic other than 2, the well-covered dimension of this graph is 2.

As shown, much work has been done on formulae for calculating the well-covered dimension of graphs. However, it is only very recently that the focus of some mathematicians has shifted from the graphs themselves, to attempting to understand geometric objects through their associated graphs. The question remains: what does the well-covered dimension of these associated graphs tell us about the geometric objects themselves? This is what Hauschild, Ortiz, and Vega were studying in [8]. They discussed the well-covered dimension of the Levi graphs of certain configurations. A configuration \((v_r, b_k)\) is a family of points and lines, where \(v\) is the number of points, \(b\) is the number of lines, there are exactly \(k\) points incident with each
line, and there are exactly $r$ lines incident with each point. The generalized quadrangles that we will be studying are a type of configuration, and thus are included in their result.

**Theorem 26** (Hauschild, Ortiz, and Vega [8]). The well-covered dimension of the Levi graph of a configuration $(v_r, b_k)$ is 0, whenever $r > 2$.

We want to examine the adjacency graph of incidence structures known as generalized quadrangles, so we will next define these objects.
GENERALIZED QUADRANGLES

In this section we introduce the definitions and representations of various types of generalized quadrangles. We also discuss many properties that are useful for our calculations later, and depict and describe the structure of a generalized quadrangle.

Definition and Properties

Definition 27. [11] A generalized quadrangle of order \((s, t)\), denoted by \(GQ(s, t)\), is an incidence structure \(S = (P, L, I)\) in which \(P\) and \(L\) are disjoint sets of objects called points and lines respectively, and for which \(I\) is a symmetric point-line incidence relation satisfying the following axioms:

1. Each point is incident with \(1 + t\) lines \((t \in \mathbb{N})\), and two distinct points are incident with at most one line.

2. Each line is incident with \(1 + s\) points \((s \in \mathbb{N})\) and two distinct lines are incident with at most one point.

3. If \(x\) is a point and \(\ell\) is a line not incident with \(x\), there exists a unique pair \((y, m) \in P \times L\) for which \(x \sim m \sim y \sim \ell\) (where \(\sim\) denotes the incidence relation).

The integers \(s\) and \(t\) are called the parameters of the generalized quadrangle. When \(s = t\), we will denote this common value by \(n\), and call \(n\) the order of the generalized quadrangle. We will denote these generalized quadrangles as \(GQ(n)\), and they will be the focus of our study.
Generalized quadrangles of order $n$ are one type of $GQ$, and we will study these specifically because many properties of generalized quadrangles are simplified when the parameters have the same value.

From this point on, we will use the more colloquial language, ‘a point is on a line’, and ‘a line contains a point’, rather than continuing the use of incidences.

The third axiom in Definition 27 can be restated to say that there are no triangles in the points and lines of $GQ(n)$. It says that for any line $\ell$ and any point $x$ not on $\ell$, there exists a unique line $m$ through the point $x$, connecting $x$ with $\ell$, at a unique point $y$. That is, there is only one line through the point $x$ that intersects $\ell$. If there were a triangle within the structure of $GQ(n)$, there would have to be a point such that there are two lines through that point, both of which intersect some other given line. Since there exists a unique such line, we can never have the two lines required to make a triangle.

The smallest possible $GQ(n)$ has order two. It is unique, and is represented pictorially in Figure 9.
This generalized quadrangle has 15 points and 15 lines. The picture in Figure 9 is important because we will use such a representation to study generalized quadrangles of larger orders. We will describe this structure in the next section.

$GQ(2)$ can also be represented by the picture in Figure 10. This representation is generally called the doily, and the points and lines are the same as the previous picture, only drawn in a different manner\(^1\)

\[\text{Figure 10. The traditional picture of } GQ(2), \text{ the doily}\]

Trace and Span in a Generalized Quadrangle

The following definitions and exposition follow Batten [1]. The reader is referred there and to the work of Payne and Thas [11] and De Bruyn and Payne [7] for more information.

Let $GQ(n) = (P, L, I)$ where $P$ is the set of all points and $L$ is the set of all lines.

The following theorem contains properties of generalized quadrangles that are well-known within the field. The reader is referred to Batten (see [1], pages 114-116) for the proof.

**Theorem 28.** Let $v = |P|$ and $b = |L|$. Let $p$ and $q$ be non-collinear points. Then

- There are precisely $t + 1$ points collinear with both $p$ and $q$,
- $v = (s + 1)(st + 1)$ and $b = (t + 1)(st + 1)$,
- $s + t$ divides $st(s + 1)(t + 1)$.

Next we define some important properties of points in generalized quadrangles.

**Definition 29.** The perp of a point is defined as $p^\perp = \{x \in P : x \sim p\}$, where $\sim$ denotes collinearity. Note that $p \in p^\perp$.

**Definition 30.** The trace of a pair of points $p \neq q \in GQ(n)$ is defined as $\{p, q\}^\perp = \{x \in P : x \sim p$ and $x \sim q\}$. If $p \sim q$, then $\{p, q\}^\perp$ is the set of points on the line $pq$.

**Definition 31.** The span of $p$ and $q$ is defined as
\( \{p, q\}^\perp = \{ x \in P : x \sim y \ \forall \ y \in \{p, q\}^\perp \} \).

The following theorem is also well-known, and the proof can be found in Batten (see [1], pages 121-122). The counting arguments follow directly from the definition of a generalized quadrangle.

**Theorem 32.** Let \( p, q \in GQ(n) \).

- For any \( p \in GQ(n) \), \( |p^\perp| = n^2 + n + 1 \).
- For any \( \{p, q\} \in GQ(n) \), \( |\{p, q\}^\perp| = n + 1 \)
- \( p, q \in \{p, q\}^\perp \), and if \( p \sim q \), then \( \{p, q\}^\perp = pq \).

In any generalized quadrangle, \( |\{p, q\}^\perp| \leq n + 1 \) for all \( p \sim q \).

**Definition 33.** The pair \( (p, q) \) is said to be regular if \( |\{p, q\}^\perp| = n + 1 \).

**Definition 34.** A point \( p \) is regular if \( (p, q) \) is regular for all \( p \neq q \).

For \( GQ(n) \), if \( (p, q) \) is a regular pair, then \( (x, y) \) is a regular pair for all \( x, y \in \{p, q\}^\perp \).

The pair \( (p, q), p \neq q \) is said to be antiregular if \( |x^\perp \cap \{p, q\}^\perp| \leq 2 \) for all \( x \in P \setminus \{p, q\} \).

**Definition 35.** The dual of an incidence structure is the structure obtained by interchanging the points and lines of the figure.

The dual of a generalized quadrangle \( GQ(s, t) \) is a generalized quadrangle of order \( (t, s) \).

In a generalized quadrangle with \( n = 3 \), either every pair of points is regular, or every non-collinear pair of points is antiregular. Furthermore,
there are two unique (up to isomorphism) generalized quadrangles of order 3, and within a given one, either all pairs of points are regular, or all pairs of non-collinear points are antiregular.

**Classes of Known Generalized Quadrangles**

The following description of the three families of classical generalized quadrangles are from Payne and Thas (see [11]), and the original definitions are due to J. Tits.

When mathematicians first began to study generalized quadrangles, it was as a special case of structures called generalized $n$-gons. This work was started in 1959 by Tits. Separately, generalized quadrangles arose in 1963 in the work by Bose, who was studying partial geometries. When Tits was studying these objects, he discovered that the following three classes of objects fulfill the axioms for a generalized quadrangle, and thus can be classified as such. It was not until more recently when Payne, Thas, and others tried to ‘generalize’ these generalized quadrangles by finding other objects which also satisfy the axioms of $GQ$s, but which are not examples of the following three classes. These original three examples have come to be called the classical generalized quadrangles, as they were discovered first, by Tits in 1959. In order to define the classical generalized quadrangles, we need a few other geometric definitions.

**Definition 36.** A finite projective space is a set of points and lines such that

1. any line contains at least three points,

2. two points are on precisely one line,
3. (Veblen’s Axiom) if \( P, Q, R, \) and \( S \) are distinct points and the lines \( PQ \)
and \( RS \) intersect each other, then so do the lines through \( PR \) and \( QS \).

The definition above leaves plenty to the imagination, namely Veblen’s Axiom. A working definition for finite projective spaces follows.

**Definition 37.** Let \( V \) be an \((d + 1)\)-dimensional vector space over a field \( \mathbb{F} \).

Define \( PG(d, \mathbb{F}) \) as the geometry in which its points are given by the
one-dimensional subspaces of \( V \), its lines are given by the two-dimensional
subspaces of \( V \), and, in general, its \( k \)-dimensional projective spaces are given
by the \((k + 1)\)-dimensional subspaces of \( V \). The incidence is set-theoretical.
We say that \( PG(d, \mathbb{F}) \) is the \( d \)-dimensional projective space over \( \mathbb{F} \). In the
case the order of \( \mathbb{F} \) is \( q \) we will write \( PG(d, q) \).

**Definition 38.** A quadric \( Q \) in \( PG(d, \mathbb{F}) \) is the set of points of \( PG(d, \mathbb{F}) \)
satisfying a second degree homogeneous polynomial equation in \( d + 1 \)
variables over \( \mathbb{F} \):

\[
\sum_{i=0}^{d} \sum_{j=0}^{d} a_{ij}x_{i}x_{j} = 0
\]

Note that this polynomial is well-defined in \( PG(d, \mathbb{F}) \) because it is
homogeneous.

The three classical generalized quadrangles can be defined as follows:

1. Consider a quadric \( Q \) of \( PG(d, q) \) with \( d = 3, 4 \) or \( 5 \). Then the points of
\( Q \) together with the lines of \( Q \) form a generalized quadrangle \( Q(d, q) \)
with parameters

\[ s = q, \ t = 1, \text{ and thus } v = (q + 1)^2, \ b = 2(q + 1) \text{ when } d = 3, \]
\( s = t = q \), and thus \( v = b = (q + 1)(q^2 + 1) \) when \( d = 4 \), and
\[\begin{align*}
\text{and thus } v &= (q + 1)(q^3 + 1), \quad b = (q^2 + 1)(q^3 + 1) \text{ when } \\
d &= 5.
\end{align*}\]

For each of these cases, \( Q(d, q) \) has a different canonical equation. They are:
\[\begin{align*}
x_0x_1 + x_2x_3 &= 0 \text{ when } d = 3, \\
x_0^2 + x_1x_2 + x_3x_4 &= 0 \text{ when } d = 4, \text{ and}
\end{align*}\]
\[f(x_0, x_1) + x_2x_3 + x_4x_5 = 0 \text{ when } d = 5, \text{ (where } f \text{ is an irreducible binary quadratic form).}\]

2. Let \( H \) be a Hermitian quadric of the projective space \( PG(d, q^2) \), \( d = 3 \) or 4. That is, \( H \) has the canonical defining equation
\[x_0^{q^1} + x_1^{q^1} + \cdots + x_d^{q^1} = 0.\]

Then the points of \( H \) together with the lines of \( H \) form a generalized quadrangle \( H(d, q^2) \) with parameters
\[\begin{align*}
s &= q^2, \quad t = q, \text{ and thus } v &= (q^2 + 1)(q^3 + 1), \quad b = (q + 1)(q^3 + 1) \text{ when } \\
d &= 3, \text{ and}
\end{align*}\]
\[\begin{align*}
s &= q^2, \quad t = q^3, \text{ and thus } v &= (q^2 + 1)(q^5 + 1), \quad b = (q^3 + 1)(q^5 + 1) \text{ when } \\
d &= 4. \text{ and}
\end{align*}\]

The third classical quadrangle is \( W(n) \), which will be important to our study, and is defined as follows.
**Definition 39.** Consider the projective space $PG(3, q)$.

1. A polarity in $PG(3, q)$ is a function mapping points to planes and lines to lines in a ‘bijective’ fashion. Lines that are mapped to themselves are said to be totally isotropic.

2. A symplectic polarity of $PG(3, q)$ has the canonical equation

$$x_0 y_1 - x_1 y_0 + x_2 y_3 - x_3 y_2 = 0.$$ 

3. The points of $PG(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a $GQ W(q)$ with parameters $s = t = q, v = b = (q + 1)(q^2 + 1)$.

   Since we are studying generalized quadrangles of order $n$, from now on we will denote this third class of classical generalized quadrangles by $W(n)$.

**Theorem 40** (See Payne and Thas [11], page 37). *The generalized quadrangle $W(n)$ is isomorphic to the dual of $Q(4, n)$.***

   It is important to notice that the three classical $GQ$s can be embedded in $PG(d, q), 3 \leq d \leq 5$.

   The following theorem is part of the folklore about generalized quadrangles, and is crucial for us:

**Theorem 41.** $W(n)$ is the only generalized quadrangle of order $n$ that has all points regular.
The Structure of $GQ(n)$

In order to study $GQ(n)$, we need to be able to represent its structure in a convenient way. We will represent $GQ(n)$ as follows.

Draw one line, $\ell_1$, and plot on it $n + 1$ points, labeling them from left to right $(1,0,0), (2,0,0), \ldots, (n+1,0,0)$. We refer to these points as base points, and this line as the base line. From each of these points $(i,0,0)$, draw $n$ lines branching upward. We call these branches, using the notation: $b_{ij}$, where $i$ is the base point $(i,0,0)$ from which the line branches, and $j = 1, 2, \ldots, n$ is the number of the branch from left to right. Each group of $n$ branches from the base point $(i,0,0)$ is called a tree, denoted $T_i$.

On each branch, plot $n$ points. Label them $(i,j,k)$, where $i$ and $j$ are as above, denoting the branch, and $k$ is the position number, $k \in \{1, 2, \ldots, n\}$, where 1 is the closest position along the branch to the base point $(i,0,0)$, and $n$ is the farthest position away from the point $(i,0,0)$.

The order we are assigning to these points and lines is arbitrary. There is no concept of order within a generalized quadrangle, but this labeling is convenient for our work, in that it allows us to have a visual representation of various types of maximal independent sets within the structure.

Figure 11 shows the structure of $GQ(3)$, the second smallest $GQ(n)$. The labeling is that which was just described, for the base line, points, branches, and trees.
Figure 11. The structure of $GQ(3)$

Note that there are $n$ more lines through each point in the branches of $GQ(n)$, in addition to the branch that the point is on. These lines cross through the branches of $GQ(n)$, and each line contains one point per tree. Points in different branches of the same tree cannot be collinear because of Axiom 3 in Definition 27.

This representation is not complete, because of the missing 27 lines, and it is not an adjacency graph, though it is helpful for allowing us to get a picture of what both of those objects look like. We do not have a picture of the adjacency graph of $GQ(3)$, because we cannot get a complete picture of every line that should be in this structure. However, by investigating this structure, we can discover how certain properties affect the $GQ$. Using this structure, we will be able to find the well-covered dimension of the adjacency graph of this object. The first method we will use to investigate properties of $GQ(n)$ is a specific type of coloring.

By coloring the set of all points that are collinear with some chosen points, we can investigate the structure of the remaining (undrawn) lines.
Consider all the lines that contain points on one branch of tree 1, branch $b_{1i}$. We will color each point in $T_2, T_3, \ldots T_{n+1}$ according to which point on $b_{1i}$ it is collinear to. We will construct all the remaining lines through points $(1, i, 1), (1, i, 2), \ldots, (1, i, n)$ for some fixed $i \in \{1, 2, \ldots, n\}$. There are $n$ remaining lines through each of these points. Note that every line through $b_{1i}$ must contain a different point of tree $j$ for every $j \in \{2, \ldots, n\}$, otherwise axiom (3) of the definition of $GQ(n)$ would be violated. When we do not consider $b_{1i}$, there are $n$ remaining lines through each of the $n$ points on $b_{1i}$, for a total of $n^2$ lines. Each of these $n^2$ lines must contain a different point in each remaining tree, thus every point in tree $j, j \in \{2, \ldots, n\}$ is covered by exactly one line that goes through $b_{1i}$.

We will now label each point on $b_{1i}$ with a different color, and every remaining point will be colored according to which point on $b_{1i}$ it is collinear to. Figure 12 depicts this process.

![Figure 12. A coloring of $GQ(3)$](image-url)
Lemma 42. Let \( \ell \) be a line in \( \mathcal{GQ}(n) \) that is not \( \ell_1 \) or a branch. Given the coloring described above, \( \ell \) must pass through exactly one point in each tree, and either

1. each of the points are the same color (\( \ell \) is a line connecting points of a certain color), or

2. each of the points must be a different color.

Proof. Choose one point, \( x \), on branch \( b_{1i} \) and suppose it is colored blue. Then every point that is collinear to \( x \) in trees \( T_2, \ldots, T_{n+1} \) is also blue. There is exactly one blue point on each branch of \( T_2, \ldots, T_{n+1} \).

Choose another point, \( p \), on \( b_{1i} \) and color it, and all points collinear to it, orange. Now consider any non-blue point, \( y \), in \( T_2 \). Suppose \( y \) is orange. Note that \( x \) and \( y \) cannot be collinear.

There are still \( n - 1 \) lines through point \( y \) that have not been accounted for. Suppose there exists a line through \( y \) that contains an orange point \( z \) in \( T_j \), \( j \in \{3, \ldots, n+1\} \). However, all orange points are collinear to the unique orange point, denoted \( p \), on branch \( b_{1i} \). Thus \( p - y - z \) is a triangle, unless \( p - y - z \) is a line.

Thus a line through \( y \) cannot contain any orange point in \( T_j \), \( j \in \{3, \ldots, n+1\} \), unless that point is on the line \( p - y \). Thus the \( n - 1 \) remaining lines through \( y \) must each contain a uniquely colored, non-orange point of each remaining tree.

Similarly, if a line \( \ell \) through \( y \) is incident with a point of a certain color in any tree, it cannot contain another point of the same color. Thus every
point on \( \ell \) must be of a different color, unless \( \ell \) is a line containing points of only one color.

The next section will consider the well-covered dimension of the smallest \( GQ(n) \), which is the only \( GQ(n) \) for which we have a complete picture of the incidence.

The Well-Covered Dimension of \( A(GQ(2)) \)

Recall that \( GQ(2) \) is the smallest \( GQ(n) \); it is unique up to isomorphism, and can be represented by Figures 9 and 10. It has 15 points and 15 lines, and when we label the points 1-15, we can find all maximal independent sets of the adjacency graph associated to \( GQ(2) \).

There are 26 MIS, each of which contains either 3 or 5 vertices. Note that 3 and 5 are both congruent to 1 (mod 2), as this will be important when we try to calculate the well-covered dimension of \( GQ(n) \) in general.

The maximal independent sets of \( GQ(2) \) are: \{1, 3, 10, 11, 12\}, \{1, 3, 14\}, \{1, 4, 8, 14, 15\}, \{1, 4, 12\}, \{1, 8, 10\}, \{1, 11, 15\}, \{2, 4, 6, 12, 13\}, \{2, 4, 15\}, \{2, 5, 9, 11, 15\}, \{2, 5, 13\}, \{2, 6, 9\}, \{2, 11, 12\}, \{3, 5, 7, 13, 14\}, \{3, 5, 11\}, \{3, 7, 10\}, \{3, 12, 13\}, \{4, 6, 8\}, \{4, 13, 14\}, \{5, 7, 9\}, \{5, 14, 15\}, \{6, 7, 8, 9, 10\}, \{6, 7, 13\}, \{6, 10, 12\}, \{7, 8, 14\}, \{8, 9, 15\}, \{9, 10, 11\}. 
These MIS give the associated matrix, which reduces to:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
\end{pmatrix}
\]

The bottom ten rows contain only 0 and have been omitted due to space.

Notice the five rows that contain only 0, 2, and 4. Over any field with characteristic 2, 2 and 4 are both congruent to 0 mod 2. Thus the bottom five rows will also be null. The well-covered dimension of the adjacency graph of $GQ(2)$ changes depending on characteristic. So we have proved the following:
Theorem 43. Over any field with characteristic 2, \( wcdim(A(GQ(2))) = 5 \), and over a field with any other characteristic, \( wcdim(A(GQ(2))) = 0 \).

However, for any \( GQ(n) \) with \( n > 2 \), we cannot create the associated matrix, because we cannot list all maximal independent sets. For \( GQ(n) \), \( n > 2 \), we do not have a complete picture of the incidence structure. In order to calculate the well-covered dimension of \( GQ(n) \) in general, we will begin by investigating types of maximal independent sets that exist in its structure.

Maximal Independent Sets and Generalized Quadrangles

Consider \( GQ(n) \), and define the adjacency graph as the graph obtained by letting the points of \( GQ(n) \) be the vertices of the adjacency graph; connected by an edge if and only if the corresponding points in \( GQ(n) \) are collinear. We will denote this graph \( A(GQ(n)) \). In order to calculate the well-covered dimension of \( A(GQ(n)) \), we must first understand its maximal independent sets.

Theorem 44. Let \( GQ(n) \) be a generalized quadrangle with every point regular. Let \( p \not\sim q \). Then \( \{p, q\}^{\perp\perp} \) is a MIS of \( A(GQ(n)) \).

Proof. All points are regular, so \( |\{p, q\}^{\perp\perp}| = n + 1 \ \forall p \) and \( q \). Color \( p \) and all points collinear to \( p \) blue. Color \( q \) and all points collinear to \( q \) orange. The set of double colored points is then \( \{p, q\}^{\perp} = \{x \in P : x \sim p \text{ and } x \sim q\} \), and \( |\{p, q\}^{\perp}| = n + 1 \). Note that \( p, q \not\in \{p, q\}^{\perp} \) because \( p \) and \( q \) are not collinear.

Then \( \{p, q\}^{\perp\perp} \) is the set of all points collinear with every double colored point.
Let \( r \in \{p,q\}^{\perp\perp} \). Consider \( r \) and color it and all points collinear to it green. We know that there are exactly \( n + 1 \) points collinear with both \( r \) and \( p \), which implies that \( r \) is collinear with exactly \( n + 1 \) blue points. But \( r \) is collinear with every double colored point, so it is collinear with all the blue points in \( \{p,q\}^\perp \). So these are exactly the blue points that \( r \) is collinear with, and similarly, exactly the \( n + 1 \) orange points that \( r \) is collinear with. Thus the points in \( \{p,q\}^\perp \) are triple colored (blue, orange, green).

Let \( s \in \{p,q\}^{\perp\perp}, s \neq r \). \( s \in \{p,q\}^{\perp\perp} \) implies that \( s \) is collinear with all the double colored blue-orange points in \( \{p,q\}^\perp \). Since there are exactly \( n + 1 \) points collinear with both \( s \) and \( p \), we have that \( s \) is also collinear with exactly the \( n + 1 \) blue points in \( \{p,q\}^\perp \). Since there are exactly \( n + 1 \) points collinear with both \( s \) and \( q \), we have that \( s \) is also collinear with exactly the \( n + 1 \) orange points in \( \{p,q\}^\perp \), and since there are exactly \( n + 1 \) points collinear with both \( s \) and \( r \), we have that \( s \) is collinear with exactly the \( n + 1 \) green points in \( \{p,q\}^\perp \). So there are only quad-colored points and single colored points at this stage.

Each point in \( \{p,q\}^{\perp\perp} \) is collinear with all the points in \( \{p,q\}^\perp \). Thus, when we color each point in \( \{p,q\}^{\perp\perp} \), it will color again the points in \( \{p,q\}^\perp \) and will single-color the other points collinear to only that one point.

Thus one point \((r)\) in \( \{p,q\}^{\perp\perp} \) colored the \( n^2 + n + 1 \) points collinear to it. Another point \((s)\) colored the \( n^2 + n + 1 \) points collinear to it, but \( n + 1 \) of them were already colored, so we colored \((n^2 + n + 1) - (n + 1) = n^2 \) new points that were not previously colored.
Yet another point, similarly, colored all \( n^2 + n + 1 \) points collinear to it, but \( n + 1 \) of them were blue-orange already, so it colored

\[
(n^2 + n + 1) - (n + 1) = n^2
\]

new points as well.

Thus after we consider the first color, the other \( n \) points in \( \{p, q\}^{\perp\perp} \) will each color an additional \( n^2 \) points.

So in total we have colored \( n^2 + n + 1 \) points with the first color, and for each of the other \( n \) colors, an additional \( n^2 \) points, for a total of

\[
n(n^2) + (n^2 + n + 1) = n^3 + n^2 + n + 1 = (n^2 + 1)(n + 1)
\]

points, which is the total number of points in \( GQ(n) \). The only points that are multiple-colored are the ones in \( \{p, q\}^{\perp} \), and every other point is single-colored.

When we choose the set of points in \( \{p, q\}^{\perp\perp} \), we will be choosing the original point of each of the \( n + 1 \) colors. Obviously a MIS can only have one point of each color, and since no points in \( \{p, q\}^{\perp\perp} \) are collinear with each other, we have that \( \{p, q\}^{\perp\perp} \) is a MIS.

Now that we have that \( \{p, q\}^{\perp\perp} \) is a MIS for all non-collinear points \( p \) and \( q \), we can describe the exact structure of that MIS.

**Theorem 45.** Let \( GQ(n) \) be a generalized quadrangle with all points regular. Fix the base line. Let \( p \) and \( q \) be points on two different branches in the same tree. Then \( \{p, q\}^{\perp\perp} \) consists of one point per branch in one tree, plus one base point in a different tree.

**Proof.** Let \( p \not\sim q \) in \( GQ(n) \). If \( p \) and \( q \) are in a tree other than \( T_1 \), we can rearrange the trees so that they are in the first tree. We have that

\[
|\{p, q\}^{\perp\perp}| = n + 1, \text{ and } |\{p, q\}^{\perp}| = n + 1.
\]

Since \( p \) and \( q \) are in \( T_1 \), the base point \((1, 0, 0)\) is in \( \{p, q\}^{\perp} \). The other \( n \) points in \( \{p, q\}^{\perp} \) must be collinear
with both $p$ and $q$. Suppose that there are points in 2 trees, other than $T_1$. Consider $\{p,q\}^{\perp\perp}$, which is the set of all points collinear with everything in $\{p,q\}^\perp$. In order to be collinear with the base point $(1,0,0)$, which is in $\{p,q\}^\perp$, every point in $\{p,q\}^{\perp\perp}$ must either be on the base line or within $T_1$. However, we can have at most one point on each branch of $T_1$ in $\{p,q\}^{\perp\perp}$, because points $\{p,q\}^{\perp\perp}$ cannot be collinear with each other. If there are points of $\{p,q\}^\perp$ in more than one other tree, there will not be any more points in $\{p,q\}^{\perp\perp}$, because a point on the baseline cannot be collinear with points in $T_2$ and $T_3$.

Thus the points of $\{p,q\}^\perp$ are the base point $(1,0,0)$, and one point per branch in one other tree. Therefore the set of points collinear to everything in $\{p,q\}^{\perp\perp}$ will be the points $p$ and $q$, plus one other point per branch in $T_1$, plus the base point of the tree that $\{p,q\}^\perp$ has points in.

Example 46. Figure 13 is a representation of a MIS in $W(3)$.

![Figure 13. A maximal independent set in $W(3)$](image)

Given points $p$ and $q$ not collinear in the same tree, we have $\{p,q\}^\perp = \{a,b,c,d\}$ and $\{p,q\}^{\perp\perp} = \{p,q,r,s\}$.
In the following corollaries, let \( p \) and \( q \) be points on two distinct branches of the same tree.

**Corollary 47.** Suppose we have \( \{p, q\} \perp = \{a_1, a_2, \ldots, a_{n+1}\} \) and \( \{p, q\} \perp \perp = \{p, q, b_1, b_2, \ldots, b_{n-1}\} \). Then

\[
\{p, q\} \perp \perp = \{p, b_i\} \perp \perp = \{q, b_i\} \perp \perp = \{b_i, b_j\} \perp \perp
\]

\( \forall 1 \leq i \leq n - 1, 1 \leq j \leq n - 1, i \neq j \).

**Proof.** Recall that \( \{p, q\} \perp = \{x : x \sim p, x \sim q\} \), and \( |\{p, q\} \perp| = n + 1 \). Also, \( \{p, q\} \perp \perp = \{x : x \sim y \forall y \in \{p, q\} \perp\} \), and \( |\{p, q, r\} \perp \perp| = n + 1 \) by regularity.

\( r \in \{p, q\} \perp \perp \) means that \( r \sim y \forall y \in \{p, q\} \perp \). This implies that \( r \) is not collinear to \( p \) or \( q \), because if it were, there would be a triangle with \( r, p \) and a point in \( \{p, q\} \perp \). Define \( \{p, q\} \perp = \{y_1, y_2, \ldots, y_{n+1}\} \). Then \( r \sim y_i \) for all \( 1 \leq i \leq s + 1 \).

Note that \( p, q \in \{p, q\} \perp \perp \), so \( p \sim y_i, q \sim y_i, r \sim y_i \) for all \( y_i \in \{p, q\} \perp \) as well.

Then, noting that \( |\{p, q\} \perp \perp| = n + 1 \) and \( |\{p, q\} \perp \perp| = n + 1 \), along with the fact that \( p, q, \) and \( r \) are all collinear to the same set of \( n + 1 \) points, we must have that \( \{p, q\} \perp \perp \) as well.

**Corollary 48.** There are exactly \( n^2(n + 1) \) distinct MIS of the form \( \{p, q\} \perp \perp \) described in Theorem 45.

**Proof.** The choice of \( p \) and \( q \) in the first \( n \) branches will determine unique points in the other branches of the chosen tree, and group all these points together in \( \{p, q\} \perp \perp \). Since we have \( n \) choices for \( p \) and \( n \) choices for \( q \), we
have \( n^2 \) distinct groupings in any given tree, and \( n + 1 \) trees, for exactly \( n^2(n + 1) \) distinct maximal independent sets of this form.

The following theorem describes another type of maximal independent set of \( A(GQ(n)) \).

**Theorem 49.** Let \( GQ(n) \) have all points regular. Let \( p \not\sim q \) be in different trees. Then \( \{p, q\}^\perp \) and \( \{p, q\}^{\perp \perp} \) both consist of one point per tree, and \( \{p, q\}^{\perp \perp} \) is a MIS.

**Proof.** We have that \( \{p, q\}^{\perp \perp} \) is a MIS by Theorem 44.

Suppose \( p \) and \( q \) are non-collinear points in different trees. \n\\(|\{p, q\}^\perp| = n + 1\), and \( \{p, q\}^\perp \) consists of all points collinear to both \( p \) and \( q \).

Since there exists a unique line from \( q \) to the branch on which \( p \) is located, and we know that \( q \not\sim p \), we have a point on the same branch as \( p \) that must be in \( \{p, q\}^\perp \). Similarly, there is a unique point on the same branch as \( q \) that must be in \( \{p, q\}^\perp \).

Note that there are \( n - 1 \) remaining points in \( \{p, q\}^\perp \). Suppose there exists a tree that contains 2 of these points in \( \{p, q\}^\perp \), labeled \( x \) and \( y \).

Clearly, \( x \) and \( y \) must be on different branches of the tree, otherwise there would be a triangle \( p - x - y \) in \( GQ(n) \).

Consider now \( \{p, q\}^{\perp \perp} \). \( \{p, q\}^{\perp \perp} \) consists of all the points collinear to every point in \( \{p, q\}^\perp \), and since all points are regular, we have that
\n\(|\{p, q\}^{\perp \perp}| = n + 1\). Obviously \( p \) and \( q \) are in \( \{p, q\}^{\perp \perp} \), as are an additional \( n - 1 \) points, labeled \( a_1, a_2, \ldots, a_{n-1} \).

Clearly \( a_i \) cannot be located in the same tree as \( p \) for any \( 1 \leq i \leq n - 1 \), because there is a point of \( \{p, q\}^\perp \) located in that tree, and
points in different branches of the same tree can never be collinear. Similarly, $a_i$ cannot be located in the same tree as $q$.

The point $a_i$ also cannot be located in the same tree as $x$ and $y$, because $a_i$ must be collinear to both $x$ and $y$. Denote the tree that $x$ and $y$ are in by $T_m$.

Then there are $n - 1$ points $a_i$ that must be contained in $n - 2$ different trees. So there must exist a tree that contains two points $a_k$ and $a_j$, $1 \leq k \leq n - 1$, $1 \leq j \leq n - 1$, $k \neq j$.

However, we know that $\{p, q\}^\perp\perp$ is a MIS. Consider the set of all points covered by $\{p, q\}^\perp\perp$. WLOG, we have $p$ in tree 1, $q$ in tree 2, $a_k$ and $a_j$ in tree $i$, $3 \leq i \leq n + 1$.

There can be no points of $\{p, q\}^\perp\perp$ in $T_m$ by above. There can also be no points on the baseline in $\{p, q\}^\perp\perp$ because the points in $\{p, q\}^\perp\perp$ must be collinear to every point in $\{p, q\}^\perp$, and we know that there exist 2 points of $\{p, q\}^\perp$ in different trees.

Thus there is no point in the MIS of $\{p, q\}^\perp\perp$ that covers the base point of $T_m$. This contradicts the fact that $\{p, q\}^\perp\perp$ is a MIS.

So we must have that all the $n + 1$ points of $\{p, q\}^\perp$ are in different trees.

Knowing this, now consider $\{p, q\}^\perp\perp$. Consider one tree, $T_i$, and suppose there are two points of $\{p, q\}^\perp\perp$ contained in it. Points in $\{p, q\}^\perp\perp$ must be collinear with each point of $\{p, q\}^\perp$. In order for a point of $\{p, q\}^\perp\perp$ to be collinear with the point of $\{p, q\}^\perp$ contained in that same tree, it must be on the same branch, because points within different branches of $T_i$ cannot be collinear with each other. If there are two points of $\{p, q\}^\perp\perp$ in tree $i$, they
must both be on the same branch as a point of \( \{p,q\}^\perp \), which contradicts the fact that \( \{p,q\}^{\perp\perp} \) is a MIS.

Thus, there can be at most one point of \( \{p,q\}^{\perp\perp} \) in each distinct tree, and since there are exactly \( n + 1 \) points in \( \{p,q\}^{\perp\perp} \), we must have exactly one point per tree. By above, the point of \( \{p,q\}^{\perp\perp} \) and the point of \( \{p,q\}^\perp \) within a single tree must be on the same branch of that tree.

Using Figure 14, we will describe the structure of some other maximal independent sets within the adjacency graph of \( GQ(n) \).

Consider the set of all \( \{p,q\}^{\perp\perp} \) with a base point in common. Without loss of generality, choose base point \( x_2 \).

There are exactly \( n \) MIS containing points in any given tree, plus the base point \( x_2 \), and these \( n \) MIS partition the branches of each tree into \( n \) groupings of \( n \) points, exactly 1 point per \( n \) branches.

Choose the points \( x_7 \) and \( x_{10} \). Then there must be a unique point \( p \) on \( b_{13} \) such that \( p \in \{x_7,x_{10}\}^{\perp\perp} \).
Without loss of generality, suppose that point is \( x_{11} \). There also must be a unique point \( q \) on the baseline such that \( q \in \{ x_7, x_{10} \} \). Suppose that point is \( x_2 \). Note that any two spans can overlap in at most one point, by Corollary 47 above.

The other maximal independent sets containing both \( x_2 \) and points in \( T_1 \) cannot contain any of \( \{ x_7, x_{10}, x_{11} \} \). Thus the other MIS containing \( x_2 \) and points in \( T_1 \) must be in \( \{ x_5, x_6 \} \times \{ x_8, x_9 \} \times \{ x_{12}, x_{13} \} \). The 3 MIS containing only points in \( T_1 \) plus \( x_2 \) partition \( T_1 \), in the sense that each such MIS overlaps in only \( x_2 \). The same situation will occur for any given tree, and a base point not in that tree.

The next two lemmas describe maximal independent sets of different cardinalities.

**Lemma 50.** There exist MIS of size \( 2n + 1 \) in \( W(n) \).

*Proof.* Let \( p \) and \( q \) be points in \( W(n) \). Recall that \( \{ p, q \} \) is a MIS for all \( p \not\sim q \) in the same tree, and the maximal independent set consists of precisely one point per branch in this tree (including \( p \) and \( q \)), plus one other base point. Denote the tree that \( p \) and \( q \) are in by \( T_i \), for some \( 1 \leq i \leq n + 1 \). Denote the tree which contains the base point in \( \{ p, q \} \) by \( T_j \), and note that \( i \neq j \).

Now consider \( \{ p, q \} \). Take only the points of \( \{ p, q \} \) that are in \( T_i \). We know that the MIS that consists of \( \{ p, q \} \) needs only one more point to be completed.

Consider the \( n \) points in \( \{ p, q \} \) that are in \( T_i \). Since the point on the baseline is only collinear to the points in the branches of \( T_i \) and the other base
points, we know that the $n$ points in $(\{p, q\}^\perp \cap T_i)$ must be collinear with every point in the branches of each tree except for the one that has the base point included in the original MIS.

So the only points that need to be added to $(\{p, q\}^\perp \cap T_i)$ in order to complete the new MIS must live in $T_j$ or the base line.

Since the points in $\{p, q\}^\perp$ must be collinear to all the points in $\{p, q\}^\perp$, and $\{p, q\}^\perp$ contains one point in each branch of $T_j$, we know that the points in $(\{p, q\}^\perp \cap T_i)$ cannot be collinear with more than one point on any branch of $T_j$. Thus there are $n - 1$ free points on each branch of $T_j$.

So for our new MIS we choose $(\{p, q\}^\perp \cap T_i)$, plus one free point on each branch of $T_j$. Now the only free points left are the base points of $T_k$, $1 \leq k \leq n$, $k \neq i, k \neq j$.

So also choose for the new MIS one of these free base points. The resulting set must be a MIS because we have no more free points to add, and we need every one of the points in the MIS for it to be complete.

So we have created a MIS of size $n + n + 1 = 2n + 1$. □

**Lemma 51.** For every $n \geq 3$, there exist MIS of size $3n$ in $W(n)$.

*Proof.* Let $p_1 \not\sim p_2$ in $W(n)$, such that $p_1$ and $p_2$ are in separate branches of the same tree. Then by Theorem 44, $\{p_1, p_2\}^\perp$ is a MIS, and by Theorem 45, $\{p_1, p_2\}^\perp$ consists of one point per branch in the tree that contains $p_1$ and $p_2$, as well as one base point in another tree. Without loss of generality, say that $p_1$ and $p_2$ are in $T_1$.

Consider the points in $\{p_1, p_2\}^\perp$ that are in $T_1$. There are $n$ points in both $\{p_1, p_2\}^\perp$ and $T_1$, one point per branch. We want to build a MIS using
Take \( p_1, p_2, \ldots, p_{n-1} \) for the MIS, and add to it the point \( q \), which is in the remaining branch of \( T_1 \).

We know that \( \{p_1, p_2\} \perp \) consists of one point per branch in a tree other than \( T_1 \), and the base point in \( T_1 \). Without loss of generality, say that the one point per branch are in \( T_2 \).

Consider \( p_n \), the point in \( T_1 \) and \( \{p_1, p_2\} \perp \perp \) that we are not choosing for the MIS. We know that the points in \( \{p_1, p_2\} \perp \perp \) cover every other point of \( W(n) \), because \( \{p_1, p_2\} \perp \perp \) is a MIS. We also know that the \( n + 1 \) points in \( T_2 \) in the trace of any pair of points in \( \{p_1, p_2\} \perp \perp \) must be the same \( n + 1 \) points, so those points are already covered, and nothing else in \( T_2 \) is covered by the span of any pair of points in \( \{p_1, p_2\} \perp \perp \). So, when we consider only the points \( \{p_1, \ldots, p_{n-1}\} \) in the MIS so far, there are \( n - 1 \) free points on each branch of \( T_2 \).

We also know that the MIS of \( \{p_1, p_2\} \perp \perp \) covers every point in \( W(n) \), so we have that in trees other than \( T_1 \) and \( T_2 \), the points \( \{p_1, p_2, \ldots, p_n\} = \{p_1, p_2\} \perp \perp \) cover every point, excluding the base point. So when we exclude the point \( p_n \) from the MIS, we have one free point per branch in every tree other than \( T_1 \) and \( T_2 \).

Every one of those free points is collinear to \( p_n \). There are exactly \( n + 1 \) lines through \( p_n \), including the branch in \( T_1 \) on which it is located. We will choose to add to our MIS exactly one point per each of the \( n \) lines through \( p_n \). We can make sure that we choose at least one point per remaining tree, since we are adding \( n \) points from \( n - 1 \) trees, and we know that there are free points on every branch of every remaining tree. Since we are choosing one point per line through \( p_n \), these \( n \) points are clearly not
collinear to each other, and by adding these $n$ points to the MIS, we cover all points in $T_3, T_4, \ldots, T_{n+1}$, including the base points since we have at least one point per tree.

When we add the point $q$ to the MIS, we cover all points on the branch it is located on in $T_1$, and so we have covered all points in $T_1$ as well. The only tree with free points now is $T_2$.

We cannot add any point in $\{p_1, p_2\}^\perp$ to the independent set, because they are all collinear to $p_1$. We cannot add the one point per branch that is collinear to $q$ to the independent set. As long as $n \geq 3$, we can choose one free point per branch in $T_2$ to add to the MIS. This will yield a MIS of size $3n$.

Remark 52. When we want to cover the remaining points in $T_2$, we could choose the base point instead of the last $n$ points, and this will yield another MIS of size $2n + 1$.

**Theorem 53.** For $W(n)$ with $n \geq 3$, the well-covered dimension of $A(W(n))$ over a field with any characteristic is 0.

**Proof.** Let $GQ(n) = W(n)$ with $n \geq 3$. Fix $i, 1 \leq i \leq n+1$. Let $p \not\sim q$, where $p$ is on one branch of $T_i$ and $q$ is on another branch of $T_i$. Then $\{p,q\}^\perp$ is a MIS of $A(W(n))$ by Theorem 44, and $\{p,q\}^\perp$ consists of one point per branch in one tree, plus one base point by Theorem 45.

Consider the set of well-covered weightings. The sum of the weights of one MIS must be equal to the sum of the weights of any other MIS.

Note that in the proof of Lemma 50, we are able to choose between $n - 1$ different base points for the last point of the maximal independent set. Since the sum of the weights in any MIS must be equal, this implies that the
weights of the points on the base line must be equal to each other. Since the base line is an arbitrary line in $GQ(n)$, we have that every vertex in $\mathcal{W}(n)$ must have the same weight.

We know that $\{p, q\} \perp \perp$ is a MIS of size $n + 1$ for all $p \not\sim q$. By Lemma 50, we have MIS of size $2n + 1$, and by Lemma 51, we have MIS of size $3n$. It follows that the sum of the weights of $n + 1$ vertices is the same as the sum of the weights of each $2n + 1$ and $3n$ vertices, while each vertex has the same weight. If this common weight is not zero, then $n + 1 = 2n + 1 = 3n (\mod p)$. This would imply that the characteristic $p$ of the field divides both $n$ and $n - 1$. Then $p = 1$, which is a contradiction unless the only possible weight is zero. So the weight of each vertex in $\mathcal{A}(\mathcal{W}(n))$ must be 0.

Thus the only function that keeps the sum of the weights of each maximal independent set constant is the zero function. That is, the zero function is the only well-covered weighting of $\mathcal{A}(\mathcal{W}(n))$. We conclude that $\text{wcdim}(\mathcal{A}(\mathcal{W}(n))) = 0$.

**Remark 54.** In order to find a MIS of size $3n$, as in Lemma 51, we must have that $n \geq 3$, so that we have at least 4 points per branch, including the base point. This is why the well-covered dimension of the adjacency graph of $GQ(2)$ is different over a field with characteristic 2. We cannot find MIS of size other than $n + 1 \mod n$ when $n = 2$, as seen in Theorem 43.
GENERALIZED QUADRANGLES OF ORDER \((s, s^2)\)

Recall from the definition of generalized quadrangles that \(GQ(s, s^2)\) will have \(s + 1\) points per line, and \(s^2 + 1\) lines per point. In this case, there will be a total of \((s + 1)(s^3 + 1)\) points, and \((s^2 + 1)(s^3 + 1)\) lines. We will define any set of three pairwise non-collinear points as a triad. A generalized quadrangle is called thick if \(s > 1\).

**Theorem 55** (Bose and Shrikhande [3]). For any thick \(GQ\) of order \((s, s^2)\), \(|\{x, y, z\}^\perp\| = s + 1\) for each triad of points \(\{x, y, z\}\).

A consequence of this theorem is the fact that \(|\{x, y, z\}^\perp\| \leq s + 1\).

Along with the definition of regular points from the previous section, we will introduce another definition that will be important for this class of generalized quadrangles.

The goal is to obtain results similar to those that were obtained for \(GQs\) with regular points, in the previous sections. If we want to calculate the well-covered dimension of these generalized quadrangles, we must find some of the maximal independent sets. We will consider the trace of a triad of points, similar to the process we followed above.

**Definition 56.** The triad \(\{x, y, z\}\) is 3-regular if and only if \(|\{x, y, z\}^\perp\| = s + 1\). The point \(x\) is called 3-regular if each triad containing \(x\) is 3-regular.
Theorem 57 (Payne and Thas [11]). Let $s \in \mathbb{N}$.

1. Let $S$ be a GQ of order $(s, s^2)$, $s > 1$, with $s$ odd. Then $S \cong Q(5, s)$ if and only if $S$ has a 3-regular point.

2. Let $S$ be a GQ of order $(s, s^2)$, with $s$ even. Then $S \cong Q(5, s)$ if and only if one of the following holds:

   (a) All points of $S$ are 3-regular.

   (b) $S$ has at least one 3-regular point not incident with some regular line.

Lemma 58. Let $GQ(s, s^2)$ be a generalized quadrangle in which all points are 3-regular. Let $\{p, q, r\}$ be a triad of points in $GQ(s, s^2)$. Let $t \in \{p, q, r\}^\perp$. Then $\{p, q, r\}^\perp = \{p, q, t\}^\perp$.

Proof. Recall that $\{p, q, r\}^\perp = \{x : x \sim p, x \sim q, x \sim r\}$, and

$|\{p, q, r\}^\perp| = s + 1$ by Theorem 55. Also,

$\{p, q, r\}^\perp = \{x : x \sim y \forall y \in \{p, q, r\}^\perp\}$, and $|\{p, q, r\}^\perp| = s + 1$ by 3-regularity.

Now $t \in \{p, q, r\}^\perp$ means that $t \sim y \forall y \in \{p, q, r\}^\perp$. This implies that $t$ is not collinear to $p, q$ or $r$, because if it were, there would be a triangle with $t, p$ and a point in $\{p, q, r\}^\perp$. Define $\{p, q, r\}^\perp = \{y_1, y_2, \ldots, y_{s+1}\}$. Then $t \sim y_i$ for all $1 \leq i \leq s + 1$.

Note that $p, q, r \in \{p, q, r\}^\perp$, so $p \sim y_i$, $q \sim y_i$, $r \sim y_i$ for all $y_i \in \{p, q, r\}^\perp$ as well.
Finally, noting that $|\{p, q, r\}^\perp\perp| = s + 1$ and $|\{p, q, t\}^\perp\perp| = s + 1$, along with the fact that $p, q, r,$ and $t$ are all collinear to the same set of $s + 1$ points, we must have that $\{p, q, r\}^\perp\perp = \{p, q, t\}^\perp\perp$.

\section*{Theorem 59.} \{p, q, r\}^\perp\perp is an independent set of $A(GQ(s, s^2))$, and covers only $\frac{s^4 - s^2 + 4s + 4}{2}$ points.

\begin{proof}
\{p, q, r\}^\perp\perp is a set of vertices within the adjacency graph of a generalized quadrangle. Clearly the points in \{p, q, r\}^\perp\perp cannot be collinear because each of them is collinear with every point of \{p, q, r\}. If $p$ and $q$ were collinear, there would be triangles, which cannot happen in a $GQ$.

\{p, q, r\}^\perp\perp contains $s + 1$ points. We know that each of the $s^2 + 1$ lines through $p$ must intersect exactly one line through each of the other points in \{p, q, r\}^\perp\perp, by Axiom 2 of the definition of generalized quadrangle.

Consider first $p$. Every line of $p$ is intersected by exactly one line of $x$ for all $x \in \{p, q, r\}^\perp\perp$. So every point collinear to $p$ is intersected by a line from one other point in \{p, q, r\}^\perp\perp.

Consider next $q$. Every line of $q$ is intersected by exactly one line of $x$ for all remaining $x \neq p \in \{p, q, r\}^\perp\perp$. So every line of $q$ has one free point left.

Consider next $r$. Similarly, every line of $r$ will be intersected by exactly one line of $x$, for $x \neq p, x \neq q \in \{p, q, r\}^\perp\perp$. This will leave 2 free points on every line of $r$.

So, the number of free points on the lines of the points in \{p, q, r\}^\perp\perp is $(0 + 1 + 2 + \cdots + s)$, for the $s + 1$ points in the set. We will multiply by $(s^2 - s)$, the number of lines that are not collinear to the points in \{p, q, r\}. To this, we need to add the points in \{p, q, r\}, of which there are $s + 1$, and
the points in \(\{p, q, r\}\perp\perp\), of which there are \(s + 1\). We calculate
\[
(0+1+2+\cdots+s)(s^2-s)+2(s+1) = \frac{s^2(s + 1)(s - 1)}{2} + 2(s+1) = \frac{s^4 - s^2 + 4s + 4}{2}.
\]

Thus \(\{p, q, r\}\perp\perp\) is an independent set of vertices that covers
\[
\frac{s^4 - s^2 + 4s + 4}{2}
\]
points.

Remark 60. \(\{p, q, r\}\perp\perp\) is not a maximal independent set of \(GQ(s, s^2)\). There are many points that the independent set is not covering. It is unclear at this point how many points are needed to complete \(\{p, q, r\}\perp\perp\) to a MIS.
CONCLUSIONS

When we began this problem, we had a suspicion that the well-covered dimension of the adjacency graphs would be zero, but we had to come up with a process to be able to prove it.

The first thing we were able to do was calculate $wcdim(A(GQ(2)))$, and since it changes depending on characteristic, we thought that this might happen in general.

We began by trying to find maximal independent sets in $GQ(n)$ for any $n$. This process was long and arduous, and really did not lead anywhere because we did not have enough information about what kinds of points could be collinear. When we decided to focus on generalized quadrangles with only regular points, we were able to prove that different cardinalities of maximal independent sets exist, and gain an idea of what they look like.

For a while, we were only able to find maximal independent sets with cardinalities congruent to 1 mod $n$, until we proved that there exist MIS of size $3n$. This result was vital in proving that the adjacency graphs of generalized quadrangles with regular points are anti-well-covered, and so was the statement that all points in the $GQ$ must have the same weight.

Putting all these ideas together, we were able to prove our main result, Theorem 53.

Hence, once the correct necessary hypothesis was identified, we were able to determine the structure of the adjacency graph of $GQ(n)$ in all its beauty and symmetry. We suspect that this behavior is ubiquitous, so we
expect to be able to generalize our results by considering other regularity conditions.

At this point, a few questions remain. By considering the 3-regularity property, we expect to be able to calculate the well-covered dimension of the adjacency graphs of $GQ$s of order $(s, s^2)$. By investigating the anti-regularity property, we may be able to find the well-covered dimension of the adjacency graphs of other classes of generalized quadrangles, including the generalized quadrangles obtained from quadrics. We may be able to discover more about the classical generalized quadrangles by investigating their canonical equations as well. We hope to be able to answer all these questions in the future.
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