On the Levi graph of point-line configurations

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(Communicated by Joseph A. Gallian)

We prove that the well-covered dimension of the Levi graph of a point-line configuration with \( v \) points, \( b \) lines, \( r \) lines incident with each point, and every line containing \( k \) points is equal to 0, whenever \( r > 2 \).

1. Introduction

The concept of the well-covered space of a graph was first introduced by Caro, Ellingham, Ramey, and Yuster [Caro et al. 1998; Caro and Yuster 1999] as an effort to generalize the study of well-covered graphs. Brown and Nowakowski [2005] continued the study of this object and, among other things, provided several examples of graphs featuring odd behaviors regarding their well-covered spaces. One of these special situations occurs when the well-covered space of the graph is trivial, i.e., when the graph is anti-well-covered. In this work, we prove that almost all Levi graphs of configurations in the family of the so-called \((v_r, b_k)\)-configurations (see Definition 3) are anti-well-covered.

We start our exposition by providing the following definitions and previously known results. Any introductory concepts we do not present here may be found in the books by Bondy and Murty [1976] and Grünbaum [2009].

We consider only simple and undirected graphs. A graph will be denoted by \( G = (V(G), E(G)) \), as is customary, where \( V(G) \) is the set of vertices of the graph and \( E(G) \) is the set of edges of the graph. We think of \( E(G) \) as an irreflexive symmetric relation on \( V(G) \). Two vertices of a graph are said to be adjacent if they are connected by an edge. An independent set of vertices is one in which no two vertices in the set are adjacent. If an independent set, \( M \), of a graph \( G \) is not a proper subset of any other independent set of \( G \), then \( M \) is a maximal independent set of \( G \).

Definition 1. Let \( G \) be a graph and \( F \) a field.

(1) A function \( f : V(G) \rightarrow F \) is said to be a weighting of \( G \). If the sum of all weights is constant for all maximal independent sets of \( G \), then the weighting is a well-covered weighting of \( G \).

MSC2010: primary 05B30; secondary 51E05, 51E30.

Keywords: Levi graph, maximal independent sets, configurations.
(2) The $F$-vector space consisting of all well-covered weightings of $G$ is called the well-covered space of $G$ (relative to $F$).

(3) The dimension of this vector space is called the well-covered dimension of $G$, denoted $\text{wcdim}(G, F)$.

**Remark 1.** For some graphs, the characteristic of the field $F$ makes a difference when calculating the well-covered dimension (see [Birnbaum et al. 2014] and [Brown and Nowakowski 2005]). If $\text{char}(F)$ does not cause a change in the well-covered dimension, then the well-covered dimension is denoted as $\text{wcdim}(G)$.

In order to calculate the well-covered dimension of a graph, $G$, one would generally need to find all possible maximal independent sets of $G$. However, finding all maximal independent sets is not always an easy task, as this is a known NP-complete problem.

Despite the NP-complete nature of this problem, let us assume that we have found all possible maximal independent sets of $G$. We will denote these maximal independent sets as $M_i$ for $i = 0, 1, \ldots, k - 1$. The well-covered weightings of $G$ are determined by solving a system of linear equations that arise from considering all equations of the form $M_0 = M_i$ for $i = 1, \ldots, k - 1$. We replace this system with the equivalent homogeneous one via standard operations and create an associated matrix $A_G$. Observe that the dimension of the nullspace of $A_G$ is equal to the dimension of the well-covered space of $G$. Thus,

$$\text{wcdim}(G, F) = |V(G)| - \text{rank}(A_G).$$

We now move onto another component of our work: configurations.

**Definition 2.** A (point-line) configuration is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is set of points, $\mathcal{L}$ is a set of lines, and $\mathcal{I}$ is an incidence relation between $\mathcal{P}$ and $\mathcal{L}$, that has the following properties:

1. Any two points are incident with at most one line.
2. Any two lines are incident with at most one point.

Next, there is some notation for configurations that needs to be set, as well as specific parameters that need to be established for the main result of this work.

**Definition 3.** We define a $(v_r, b_k)$-configuration as a configuration such that

1. $|\mathcal{P}| = v$, and $v \geq 4$.
2. $|\mathcal{L}| = b$, and $b \geq 4$.
3. There are exactly $k$ points incident with each line, and $k \geq 2$.
4. There are exactly $r$ lines incident with each point, and $r \geq 2$.

When $v = b$ and $r = k$, the configuration will be denoted by $(v_r)$. 
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Example 1. Several well-known geometric structures fall into the category of 
\((v, b, k)\)-configurations. For instance:

1. A projective plane of order \(q\) is a \((q^2 + q + 1)_{(q+1)}\)-configuration, where \(q\) is
the power of a prime. See Figure 1 for a representation of \(PG(2, 3) = (13_4)\).

2. The Pappus configuration is a \((9_3)\)-configuration, and the Desargues configu-
ration is a \((10_3)\)-configuration.

3. \(PG(n, q)\) is a
\[
\left( \frac{q^{n+1} - 1}{q-1}_{(q+1)}, \frac{(q^{n+1} - 1)(q^n - 1)}{(q^2 - 1)(q - 1)}_{(q^2+q+1)} \right)
\]
configuration, where \(q\) is the power of a prime.

4. A generalized quadrangle \(G(s, t)\) is a \(((1+s)(st+1)_{(1+s)}, (1+t)(st+1)_{(1+t)})\)-
configuration.

The reader is referred to the book by Batten [1997] for more information about
these important geometric objects.

Finally, we define Levi graphs, which will connect configurations and graphs.

Definition 4. The Levi graph of a \((v, b, k)\)-configuration \((P, L, I)\) is the bipartite
graph \(G\) with \(V(G) = P \cup L\) and \(E(G) = I\). That is, \(p \in P\) is adjacent to \(\ell \in L\) if
and only if \(p \mathcal{I} \ell\). We will denote this graph \(\text{Levi}(v, b, k)\).
Note that \(P\) and \(L\) are independent sets — the partite sets — in \(G\).

Our main result, which will be proven in the following section, combines all of
these objects as follows:

Theorem 1. If \(r\) is a positive integer greater than \(2\), then \(\text{wcdim}(\text{Levi}_{(v_r, b_k)}) = 0\).

We would like to remark that Theorem 1 says is that almost all Levi graphs of
\((v_r, b_k)\)-configurations are anti-well-covered.
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Figure 2. A maximal independent set $M_P$ in Levi($v_{13,4}$).

2. The well-covered dimension of Levi($v_r, b_k$)

We will prove Theorem 1 by first proving a technical lemma that introduces a family of maximal independent sets that will prove to be useful later on.

Lemma 1. A Levi graph of a configuration ($v_r, b_k$), where $r > 2$, has at least $v + b + 2$ maximal independent sets.

Proof. Let $P$ be a fixed point in ($v_r, b_k$). We consider the set, $M_P$, of vertices of Levi($v_r, b_k$) given by $P$ and all the lines not incident to $P$. This is an independent set of Levi($v_r, b_k$) because there is no incidence between vertices in the set. Moreover, note that if we included another point-vertex to $M_P$, then that vertex would be adjacent to one of the line-vertices in $M_P$ (because of condition (2) in Definition 2, and the fact that $r > 2$). Also, if another line-vertex were to be added to $M_P$, then this line would have to be incident with $P$. It follows that $M_P$ is a maximal independent set of Levi($v_r, b_k$). See Figure 2 for an example.

Repeating this construction for all $v$ points in ($v_r, b_k$), we get $v$ distinct maximal independent sets of Levi($v_r, b_k$).

We will now construct another $b$ distinct maximal independent sets of Levi($v_r, b_k$). We start by fixing a line $\ell$ in ($v_r, b_k$) and then any two distinct points $P_1, P_2 \in \ell$ (recall that $k \geq 2$). We consider the set, $M_{P_1, P_2}$ of vertices of Levi($v_r, b_k$) given by $P_1, P_2$ and all the lines not incident to either of these points. Note that this forms an independent set since adjacency in Levi($v_r, b_k$) only occurs if incidence occurs in ($v_r, b_k$). If we try to add in another vertex-point to $M_{P_1, P_2}$, since $r > 2$, this point will be incident to one of the lines not through $P_1$ or $P_2$ and will therefore be adjacent to the vertex-lines in $M_{P_1, P_2}$. If we try to add another vertex-line to $M_{P_1, P_2}$, then this line will be incident to one or both of $P_1$ and $P_2$. Therefore, $M_{P_1, P_2}$ is a maximal independent set of Levi($v_r, b_k$). See Figure 3 for an example.

Repeating this construction for all $b$ lines in ($v_r, b_k$) (it does not matter what pair of points one picks on any given line), we get $b$ distinct maximal independent sets of Levi($v_r, b_k$).
Finally, note that the set of all point-vertices in \( \text{Levi}(v_r, b_k) \) is a maximal independent set and the set of all line-vertices in \( \text{Levi}(v_r, b_k) \) is as well. Hence, we have constructed \( v + b + 2 \) distinct maximal independent sets in \( \text{Levi}(v_r, b_k) \).

Next, we proceed to prove our main result.

**Proof of Theorem 1.** We denote by \( F \) the field of scalars of the well-covered space of \( G = \text{Levi}(v_r, b_k) \), where \( r > 2 \). Let \( A_G \) be the associated matrix of \( G \), and note that \( A_G \) has \( v + b \) columns. In order to prove that \( A_G \) has \( v + b \) linearly independent rows we will consider the \( v + b + 2 \) maximal independent sets in Lemma 1.

We create the first \( v \) rows of \( A_G \) by equating the weight of each of the maximal independent sets \( M_P \) to the weight of the maximal independent set consisting of all the lines of \( G \). After subtracting, we obtain \( v \) equations of the form

\[
f(P) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) = 0,
\]

where each \( \ell_i \) is incident with \( P \). It follows that, after organizing the columns of \( A_G \) by putting point-vertices first and then line-vertices, the “first” \( v \) rows of \( A_G \) are

\[
\begin{bmatrix}
I_v & -C
\end{bmatrix},
\]

where \( C \) is the incidence matrix of \( \text{Levi}(v_r, b_k) \).

In order to obtain the next \( b \) rows of \( A_G \), we will consider maximal independent sets of the form \( M_{P,Q} \). For any given line \( \ell \) of \( (v_r, b_k) \), we choose (any) two points on it. We will denote these two points as \( P_1 \) and \( P_2 \). We then consider the maximal independent set \( M_{P_1,P_2} \) and equate its weight to the weight of the maximal independent set \( M_{P_1} \). After subtracting, we obtain an equation of the form

\[
f(P_2) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) + f(\ell) = 0,
\]

where each \( \ell_i \) is incident with \( P_2 \).

Note that subtracting (1) (with \( P = P_2 \)) from (2) yields \( f(\ell) = 0 \). Since \( \ell \) was arbitrary, we get \( f(\ell) = 0 \) for every line in \( (v_r, b_k) \). It follows that since subtracting...
equations is just a different way to describe row operations in $A_G$, we get that the “first" $v + b$ rows of $A_G$ (after a few row operations) are

$$\begin{bmatrix}
i_v & -C \\
0 & I_b
\end{bmatrix}.$$ 

Note that addition and subtraction were the only two (row) operations needed to obtain the matrix above. Hence, the first $v + b$ rows of $A_G$ do not change depending on the characteristic of $F$.

Since the determinant of the matrix above is nonzero, the rank of $A_G$ is maximal, and thus $\text{wcdim}(\text{Levi}(v_r, b_k)) = 0$. \hfill \square

3. Possible generalizations

In this section, we study possible generalizations of Theorem 1. This will be done by providing a few results and by introducing objects to which this theorem could be extended. We begin by proving that Theorem 1 cannot be extended to configurations having exactly two lines being incident with every point. This will be done by an example that considers $(v_2)$-configurations.

We first notice that a $(v_2)$-configuration is a disjoint union of polygons/cycles. This is convenient because disjoint unions of graphs behave well with respect to the well-covered dimension. In fact, Lemma 5 in [Brown and Nowakowski 2005] says

$$\text{wcdim}(G \cup H) = \text{wcdim}(G) + \text{wcdim}(H),$$

where $\cup$ stands for disjoint union.

Since we know that Levi$_{C_n} = C_{2n}$, we get the following lemma.

**Lemma 2.** Let $C$ be a $(v_2)$-configuration. Then,

$$C = \bigcup_{i=1}^{t} C_{n_i},$$

where $n_i > 2$, for all $1 \leq i \leq t$. Moreover,

$$\text{wcdim}(\text{Levi}_C) = \sum_{i=1}^{t} \text{wcdim}(C_{2n_i}).$$

Finally, we notice that Theorem 5 in [Birnbaum et al. 2014] implies

$$\text{wcdim}(C_{2n}) = \begin{cases} 
2 & \text{if } n = 3, \\
0 & \text{if } n \geq 4.
\end{cases}$$

Next is an immediate corollary of that same theorem, together with our Lemma 2.
**Corollary 1.** The well-covered dimension of $\text{Levi}_C$ is even for all $(v_2)$-configurations $C$. Moreover, for every $n \in \mathbb{N}$, there is a $(v_2)$-configuration, $C_n$, such that

$$\text{wcdim}(\text{Levi}_{C_n}) = 2n.$$  

In particular, the sequence $\{\text{wcdim}(\text{Levi}_{C_n})\}_{n=1}^{\infty}$ is unbounded.

We conclude that Theorem 1 cannot be expanded to the case $r = 2$. However, it is still an open problem to find the well-covered dimension of all Levi graphs of $(v_2, b_k)$-configurations.

Of course, the study of the well-covered dimension of Levi graphs of configurations not of the form $(v_r, b_k)$ is also an interesting open problem.

Block designs are another family of objects that could be studied to attempt a generalization of Theorem 1. These objects can be much less “geometric” than $(v_r, b_k)$-configurations, given that they are obtained after relaxing items (3) and (4) in Definition 2. In order to be more precise, we provide the following definition.

**Definition 5.** Let $\lambda, t \geq 1$. A $t$-$(v, k, \lambda)$-design (or $t$-design), is an incidence structure of points and blocks with the following properties:

1. There are $v$ points.
2. Each block is incident with $k$ points.
3. Any $t$ points are incident with $\lambda$ common blocks.

It is easy to see that a 1-$(v, k, \lambda)$-design is a $(v_\lambda, b_k)$-configuration, where $b = v\lambda/k$. Moreover, a 2-$(v, k, 1)$-design is a configuration in which every pair of points are “collinear”. For $t > 1$ and $\lambda > 1$, the obvious definition of the Levi graph of a $t$-design would yield a multigraph. This apparent setback is not so much of a problem since having one edge or multiple edges between two vertices would mean the same thing when looking for maximal independent sets. We claim that the ideas used to prove Theorem 1 can be generalized to be applicable to block designs.

Finally, in this work, we studied the well-covered space of the Levi graph of a $(v_r, b_k)$-configuration. We propose, as an interesting open problem, the study of configurations via understanding the well-covered spaces of their collinearity graphs (in which points in a configuration are defined as vertices, and adjacency occurs if and only if the points are collinear). The third author is currently working on a particular case of this problem: generalized quadrangles.

**Acknowledgments**

The authors would like to thank the referee for his/her suggestions. We believe they greatly improved the exposition in this article. We gratefully acknowledge the NSF for their financial support (Grant #DMS-1156273), and the REU program at California State University, Fresno.
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Received: 2014-10-04 Revised: 2014-12-28 Accepted: 2015-01-02

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